Operator $\ell_p \to \ell_q$ norms of random matrices with iid entries

Marta Strzelecka

(based on joint work in progress with Rafał Latała)

University of Warsaw

AGA seminar

October 12, 2023

Setting:

- $1 \le p, q \le \infty$;
- $\|x\|_p = (\sum_{j=1}^n |x_j|^p)^{1/p}$ (ℓ_p-norm in $\mathbb{R}^n$);
- p^* – Hölder's conjugate: $\frac{1}{p} + \frac{1}{p^*} = 1$, so $\|\cdot\|_{p^*}$ is the dual norm of $\|\cdot\|_p$;
- B_p^n is the unit ball of ℓ_p norm in $\mathbb{R}^n$;
- $\|x\|_{p^*} = \sup_{t \in B_p^n} \sum_{j=1}^n t_j x_j$;
- consider a random $m \times n$ matrix $X = (X_{i,j})_{i \le m, j \le n}$ with iid entries $X_{i,j}$.

Setting:

- $1 \leq p, q \leq \infty$;
- $\|x\|_p = (\sum_{j=1}^n |x_j|^p)^{1/p}$ (ℓ_p-norm in $\mathbb{R}^n$);
- p^* – Hölder's conjugate: $\frac{1}{p} + \frac{1}{p^*} = 1$, so $\|\cdot\|_{p^*}$ is the dual norm of $\|\cdot\|_p$;
- B_p^n is the unit ball of ℓ_p norm in $\mathbb{R}^n$;
- $\|x\|_{p^*} = \sup_{t \in B_p^n} \sum_{j=1}^n t_j x_j$;
- consider a random $m \times n$ matrix $X = (X_{i,j})_{i \leq m, j \leq n}$ with iid entries $X_{i,j}$.

Aim:

$$\mathbb{E}\|X\|_{\ell_p^n \to \ell_q^m} := \mathbb{E}\|X \colon \ell_p^n \to \ell_q^m\| \sim \quad ?$$

$f \lesssim g$ means $f \leq Cg$, $f \sim g$ means $\frac{1}{C} g \leq f \leq Cg$.

Setting:

- $1 \le p, q \le \infty$;
- $\|x\|_p = (\sum_{j=1}^n |x_j|^p)^{1/p}$ (ℓ_p-norm in $\mathbb{R}^n$);
- p^* – Hölder's conjugate: $\frac{1}{p} + \frac{1}{p^*} = 1$, so $\|\cdot\|_{p^*}$ is the dual norm of $\|\cdot\|_p$;
- B_p^n is the unit ball of ℓ_p norm in $\mathbb{R}^n$;
- $\|x\|_{p^*} = \sup_{t \in B_p^n} \sum_{j=1}^n t_j x_j$;
- consider a random $m \times n$ matrix $X = (X_{i,j})_{i \le m, j \le n}$ with iid entries $X_{i,j}$.

Aim:

$$\mathbb{E}\|X\|_{\ell_p^n \to \ell_q^m} := \mathbb{E}\|X \colon \ell_p^n \to \ell_q^m\| \sim \quad ?$$

$f \lesssim g$ means $f \le Cg$, $f \sim g$ means $\frac{1}{C} g \le f \le Cg$.

$$\mathbb{E}\|X\|_{\ell_p^n \to \ell_q^m} = \mathbb{E} \sup_{s \in B_{q^*}^m, t \in B_p^n} \sum_{i,j} X_{i,j} s_i t_j$$

Setting:

- $1 \leq p, q \leq \infty$;
- $\|x\|_p = (\sum_{j=1}^n |x_j|^p)^{1/p}$ (ℓ_p-norm in $\mathbb{R}^n$);
- p^* – Hölder's conjugate: $\frac{1}{p} + \frac{1}{p^*} = 1$, so $\|\cdot\|_{p^*}$ is the dual norm of $\|\cdot\|_p$;
- B_p^n is the unit ball of ℓ_p norm in $\mathbb{R}^n$;
- $\|x\|_{p^*} = \sup_{t \in B_p^n} \sum_{j=1}^n t_j x_j$;
- consider a random $m \times n$ matrix $X = (X_{i,j})_{i \leq m, j \leq n}$ with iid entries $X_{i,j}$.

Aim:

$$\mathbb{E}\|X\|_{\ell_p^n \to \ell_q^m} := \mathbb{E}\|X \colon \ell_p^n \to \ell_q^m\| \sim \quad ?$$

$f \lesssim g$ means $f \leq Cg$, $f \sim g$ means $\frac{1}{C} g \leq f \leq Cg$.

$$\mathbb{E}\|X\|_{\ell_p^n \to \ell_q^m} = \mathbb{E} \sup_{s \in B_{q^*}^m, t \in B_p^n} \sum_{i,j} X_{i,j} s_i t_j$$

Duality $(p^*, q, n, m) \longleftrightarrow (q, p^*, m, n)$.

In the Gaussian case we may apply the Chevet inequality:

$$\mathbb{E} \sup_{s\in S, t\in T} \sum_{i\le m, j\le n} g_{i,j} s_i t_j$$

$$\sim \sup_{s\in S} \|s\|_2 \, \mathbb{E} \sup_{t\in T} \sum_{j=1}^{n} g_j t_j + \sup_{t\in T} \|t\|_2 \, \mathbb{E} \sup_{s\in T} \sum_{i=1}^{m} g_i s_i .$$

In the Gaussian case we may apply the Chevet inequality:

$$\mathbb{E} \sup_{s\in S, t\in T} \sum_{i\le m, j\le n} g_{i,j} s_i t_j$$

$$\sim \sup_{s\in S} \|s\|_2 \, \mathbb{E} \sup_{t\in T} \sum_{j=1}^{n} g_j t_j + \sup_{t\in T} \|t\|_2 \, \mathbb{E} \sup_{s\in T} \sum_{i=1}^{m} g_i s_i.$$

In the Gaussian case we may apply the Chevet inequality:

$$\mathbb{E} \sup_{s \in B_{q^*}^m, t \in B_p^n} \sum_{i \le m, j \le n} g_{i,j} s_i t_j$$

$$\sim \sup_{s \in B_{q^*}^m} \|s\|_2 \, \mathbb{E} \sup_{t \in B_p^n} \sum_{j=1}^{n} g_j t_j + \sup_{t \in B_p^n} \|t\|_2 \, \mathbb{E} \sup_{s \in B_{q^*}^m} \sum_{i=1}^{m} g_i s_i .$$

In the Gaussian case we may apply the Chevet inequality:

$$\mathbb{E} \sup_{s\in B^m_{q^*}, t\in B^n_p} \sum_{i\le m, j\le n} g_{i,j} s_i t_j \sim \sup_{s\in B^m_{q^*}} \|s\|_2 \, \mathbb{E}\|(g_j)_{j\le n}\|_{p^*} + \sup_{t\in B^n_p} \|t\|_2 \, \mathbb{E}\|(g_i)_{i\le m}\|_q .$$

In the Gaussian case we may apply the Chevet inequality:

$$\mathbb{E} \sup_{s\in B^m_{q^*}, t\in B^n_p} \sum_{i\le m, j\le n} g_{i,j} s_i t_j$$
$$\sim \sup_{s\in B^m_{q^*}} \|s\|_2 \,\mathbb{E}\|(g_j)_{j\le n}\|_{p^*} + \sup_{t\in B^n_p} \|t\|_2 \,\mathbb{E}\|(g_i)_{i\le m}\|_q.$$

Corollary:

$$\mathbb{E}\left\|(g_{i,j})_{i\le m, j\le n}\right\|_{\ell^n_p\to\ell^m_q}$$
$$\sim \begin{cases} m^{1/q-1/2} n^{1/p^*} + n^{1/p^*-1/2} m^{1/q}, & p^*, q\le 2,\\ \sqrt{p^*\wedge \operatorname{Log} n}\, n^{1/p^*} m^{1/q-1/2} + m^{1/q}, & q\le 2\le p^*,\\ n^{1/p^*} + \sqrt{q\wedge \operatorname{Log} m}\, m^{1/q} n^{1/p^*-1/2}, & p^*\le 2\le q,\\ \sqrt{p^*\wedge \operatorname{Log} n}\, n^{1/p^*} + \sqrt{q\wedge \operatorname{Log} m}\, m^{1/q}, & 2\le q, p^* \end{cases}$$

$$a\vee b = \max\{a,b\}, \qquad a\wedge b = \min\{a,b\},$$
$$\operatorname{Log} n = 1\vee \ln n$$

In the Gaussian case we may apply the Chevet inequality:

$$\mathbb{E} \sup_{s\in B^m_{q^*}, t\in B^n_p} \sum_{i\le m, j\le n} g_{i,j} s_i t_j \sim \sup_{s\in B^m_{q^*}} \|s\|_2 \, \mathbb{E}\|(g_j)_{j\le n}\|_{p^*} + \sup_{t\in B^n_p} \|t\|_2 \, \mathbb{E}\|(g_i)_{i\le m}\|_q .$$

Corollary:

$$\mathbb{E}\big\|(g_{i,j})_{i\le m, j\le n}\big\|_{\ell^n_p\to\ell^m_q} \sim \begin{cases} m^{1/q-1/2} n^{1/p^*} + n^{1/p^*-1/2} m^{1/q}, & p^*, q\le 2,\\ \sqrt{p^*\wedge \operatorname{Log} n}\, n^{1/p^*} m^{1/q-1/2} + m^{1/q}, & q\le 2\le p^*,\\ n^{1/p^*} + \sqrt{q\wedge \operatorname{Log} m}\, m^{1/q} n^{1/p^*-1/2}, & p^*\le 2\le q,\\ \sqrt{p^*\wedge \operatorname{Log} n}\, n^{1/p^*} + \sqrt{q\wedge \operatorname{Log} m}\, m^{1/q}, & 2\le q, p^* \end{cases}$$

What if $X_{i,j}$ are exponential? Or, more generally, symmetric Weibull r.v.'s with parameter $r\in[1,2]$:

$$\mathbb{P}(|X_i|\ge t) = e^{-t^r} \quad ?$$

Theorem (Latała–S., 2023)

Let $X_{i,j}$, X_i, X_j, $1 \le i \le m$, $1 \le j \le n$ be iid symmetric Weibull r.v.'s with parameter $r \in [1,2]$. Then

$$
\begin{aligned}
&\mathbb{E} \sup_{s\in S, t\in T} \sum_{i\le m, j\le n} X_{i,j} s_i t_j \\
&\sim \sup_{s\in S} \|s\|_{r^*} \mathbb{E} \sup_{t\in T} \sum_{j=1}^{n} X_j t_j + \sup_{s\in S} \|s\|_2 \mathbb{E} \sup_{t\in T} \sum_{j=1}^{n} g_j t_j \\
&\qquad + \sup_{t\in T} \|t\|_{r^*} \mathbb{E} \sup_{s\in S} \sum_{i=1}^{m} X_i s_i + \sup_{t\in T} \|t\|_2 \mathbb{E} \sup_{s\in S} \sum_{i=1}^{m} g_i s_i \\
&\sim \sup_{s\in S} \|s\|_{r^*} \mathbb{E} \sup_{t\in T} \sum_{j=1}^{n} X_j t_j + \sup_{t\in T} \|t\|_{r^*} \mathbb{E} \sup_{s\in S} \sum_{i=1}^{m} X_i s_i \\
&\qquad + \mathbb{E} \sup_{s\in S, t\in T} \sum_{i\le m, j\le n} g_{i,j} s_i t_j.
\end{aligned}
$$

$$
\mathbb{E}\big\|(X_{i,j})_{i\le m, j\le n}\big\|_{\ell_p^n\to\ell_q^m}
\sim \begin{cases}
m^{1/q-1/2}n^{1/p^*}+n^{1/p^*-1/2}m^{1/q}, & p^*,q\le 2,\\
(p^*\wedge \operatorname{Log} n)^{1/r}n^{1/p^*}m^{(1/q-1/r)\vee 0} & \\
\quad +\sqrt{p^*\wedge \operatorname{Log} n}\, n^{1/p^*}m^{1/q-1/2}+m^{1/q}, & q\le 2\le p^*,\\
n^{1/p^*}+(q\wedge \operatorname{Log} m)^{1/r}m^{1/q}n^{(1/p^*-1/r)\vee 0} & \\
\quad +\sqrt{q\wedge \operatorname{Log} m}\, m^{1/q}n^{1/p^*-1/2}, & p^*\le 2\le q,\\
(p^*\wedge \operatorname{Log} n)^{1/r}n^{1/p^*}+(q\wedge \operatorname{Log} m)^{1/r}m^{1/q}, & 2\le p^*,q.
\end{cases}
$$

$$
\mathbb{E}\big\|(X_{i,j})_{i\le m, j\le n}\big\|_{\ell_p^n\to\ell_q^m}
\sim \begin{cases}
m^{1/q-1/2}n^{1/p^*} + n^{1/p^*-1/2}m^{1/q}, & p^*, q\le 2,\\
(p^*\wedge \operatorname{Log} n)^{1/r}n^{1/p^*}m^{(1/q-1/r)\vee 0} & \\
\quad +\sqrt{p^*\wedge \operatorname{Log} n}\, n^{1/p^*}m^{1/q-1/2} + m^{1/q}, & q\le 2\le p^*,\\
n^{1/p^*} + (q\wedge \operatorname{Log} m)^{1/r}m^{1/q}n^{(1/p^*-1/r)\vee 0} & \\
\quad +\sqrt{q\wedge \operatorname{Log} m}\, m^{1/q}n^{1/p^*-1/2}, & p^*\le 2\le q,\\
(p^*\wedge \operatorname{Log} n)^{1/r}n^{1/p^*} + (q\wedge \operatorname{Log} m)^{1/r}m^{1/q}, & 2\le p^*, q.
\end{cases}
$$

$$
\mathbb{E}\big\|(g_{i,j})_{i\le m, j\le n}\big\|_{\ell_p^n\to\ell_q^m}
\sim \begin{cases}
m^{1/q-1/2}n^{1/p^*} + n^{1/p^*-1/2}m^{1/q}, & p^*, q\le 2,\\
\sqrt{p^*\wedge \operatorname{Log} n}\, n^{1/p^*}m^{1/q-1/2} + m^{1/q}, & q\le 2\le p^*,\\
n^{1/p^*} + \sqrt{q\wedge \operatorname{Log} m}\, m^{1/q}n^{1/p^*-1/2}, & p^*\le 2\le q,\\
\sqrt{p^*\wedge \operatorname{Log} n}\, n^{1/p^*} + \sqrt{q\wedge \operatorname{Log} m}\, m^{1/q}, & 2\le q, p^*.
\end{cases}
$$

$$
\begin{aligned}
&\mathbb{E}\big\|(X_{i,j})_{i\le m, j\le n}\big\|_{\ell_p^n\to\ell_q^m} \\
&\sim \begin{cases}
m^{1/q-1/2}n^{1/p^*} + n^{1/p^*-1/2}m^{1/q}, & p^*, q \le 2, \\
(p^*\wedge \operatorname{Log} n)^{1/r} n^{1/p^*} m^{(1/q-1/r)\vee 0} & \\
\quad +\sqrt{p^*\wedge \operatorname{Log} n}\, n^{1/p^*} m^{1/q-1/2} + m^{1/q}, & q \le 2 \le p^*, \\
n^{1/p^*} + (q\wedge \operatorname{Log} m)^{1/r} m^{1/q} n^{(1/p^*-1/r)\vee 0} & \\
\quad +\sqrt{q\wedge \operatorname{Log} m}\, m^{1/q} n^{1/p^*-1/2}, & p^* \le 2 \le q, \\
(p^*\wedge \operatorname{Log} n)^{1/r} n^{1/p^*} + (q\wedge \operatorname{Log} m)^{1/r} m^{1/q}, & 2 \le p^*, q.
\end{cases}
\end{aligned}
$$

The Chevet-type inequality also yields two-side bounds for

$$
\mathbb{E}\big\|(a_{i,j}X_{i,j})_{i\le m, j\le n}\big\|_{\ell_p^n\to\ell_q^m},
$$

where $a_{i,j}$'s are deterministic.

What happens when $X_{i,j} = \varepsilon_{i,j}$ (iid random signs)? "$r = \infty$"

What happens when $X_{i,j} = \varepsilon_{i,j}$ (iid random signs)? "$r = \infty$"

Theorem (Bennett–Goodman–Newman, 1975)

$$\mathbb{E}\|X\colon \ell_2^n \to \ell_q^m\| \lesssim_q \max\{n^{1/2}, m^{1/q}\}.$$

What happens when $X_{i,j} = \varepsilon_{i,j}$ (iid random signs)? $\quad$ "$r = \infty$"

Theorem (Bennett–Goodman–Newman, 1975)

$$\mathbb{E}\|X\colon \ell_2^n \to \ell_q^m\| \lesssim_q \max\{n^{1/2}, m^{1/q}\}.$$

Corollary

$$\mathbb{E}\big\|(\varepsilon_{i,j})_{i\le m, j\le n}\big\|_{\ell_p^n \to \ell_q^m} \sim_{p,q} \begin{cases} m^{1/q-1/2}n^{1/p^*} + n^{1/p^*-1/2}m^{1/q}, & p^*, q \le 2,\\ m^{1/q-1/2}n^{1/p^*} + m^{1/q}, & q \le 2 \le p^*,\\ n^{1/p^*} + n^{1/p^*-1/2}m^{1/q}, & p^* \le 2 \le q,\\ n^{1/p^*} + m^{1/q}, & 2 \le p^*, q. \end{cases}$$

What happens when $X_{i,j} = \varepsilon_{i,j}$ (iid random signs)? "$r = \infty$"

Theorem (Bennett–Goodman–Newman, 1975)

$$\mathbb{E}\|X\colon \ell_2^n \to \ell_q^m\| \lesssim_q \max\{n^{1/2}, m^{1/q}\}.$$

Corollary

$$\mathbb{E}\left\|(\varepsilon_{i,j})_{i\le m, j\le n}\right\|_{\ell_p^n\to\ell_q^m} \sim_{p,q} \begin{cases} m^{1/q-1/2}n^{1/p^*} + n^{1/p^*-1/2}m^{1/q}, & p^*, q \le 2, \\ m^{1/q-1/2}n^{1/p^*} + m^{1/q}, & q \le 2 \le p^*, \\ n^{1/p^*} + n^{1/p^*-1/2}m^{1/q}, & p^* \le 2 \le q, \\ n^{1/p^*} + m^{1/q}, & 2 \le p^*, q. \end{cases}$$

What is the dependence on p and q?

$$
\begin{aligned}
&\mathbb{E}\big\|(\varepsilon_{i,j})_{i\le m, j\le n}\big\|_{\ell_p^n\to\ell_q^m} \\
&\quad\sim \begin{cases}
m^{1/q-1/2}n^{1/p^*} + n^{1/p^*-1/2}m^{1/q}, & p^*, q\le 2,\\
\sqrt{p^*\wedge m}\, n^{1/p^*}m^{1/q-1/2} + m^{1/q}, & q\le 2\le p^*,\\
n^{1/p^*} + \sqrt{q\wedge n}\, m^{1/q}n^{1/p^*-1/2}, & p^*\le 2\le q,\\
(p^*\wedge m)^{1/q}n^{1/p^*} + (q\wedge n)^{1/p^*}m^{1/q}, & 2\le p^*, q.
\end{cases}
\end{aligned}
$$

$$\mathbb{E}\left\|(\varepsilon_{i,j})_{i\le m, j\le n}\right\|_{\ell_p^n\to\ell_q^m} \sim \begin{cases} m^{1/q-1/2}n^{1/p^*} + n^{1/p^*-1/2}m^{1/q}, & p^*, q\le 2,\\ \sqrt{p^*\wedge m}\, n^{1/p^*}m^{1/q-1/2} + m^{1/q}, & q\le 2\le p^*,\\ n^{1/p^*} + \sqrt{q\wedge n}\, m^{1/q}n^{1/p^*-1/2}, & p^*\le 2\le q,\\ (p^*\wedge m)^{1/q}n^{1/p^*} + (q\wedge n)^{1/p^*}m^{1/q}, & 2\le p^*, q. \end{cases}$$

$$\mathbb{E}\left\|(g_{i,j})_{i\le m, j\le n}\right\|_{\ell_p^n\to\ell_q^m} \sim \begin{cases} m^{1/q-1/2}n^{1/p^*} + n^{1/p^*-1/2}m^{1/q}, & p^*, q\le 2,\\ \sqrt{p^*\wedge \operatorname{Log} n}\, n^{1/p^*}m^{1/q-1/2} + m^{1/q}, & q\le 2\le p^*,\\ n^{1/p^*} + \sqrt{q\wedge \operatorname{Log} m}\, m^{1/q}n^{1/p^*-1/2}, & p^*\le 2\le q,\\ \sqrt{p^*\wedge \operatorname{Log} n}\, n^{1/p^*} + \sqrt{q\wedge \operatorname{Log} m}\, m^{1/q}, & 2\le q, p^*. \end{cases}$$

In the case when $X_{i,j}$ are Weibulls with $r \in [1,2]$ or random signs:

$$\mathbb{E}\left\|(X_{i,j})_{i\le m, j\le n}\right\|_{\ell_p^n\to\ell_q^m} \sim m^{1/q}\sup_{t\in B_p^n}\left\|\sum_{j=1}^n t_j X_{1,j}\right\|_{q\wedge \mathrm{Log}\, m} + n^{1/p^*}\sup_{s\in B_{q^*}^m}\left\|\sum_{i=1}^m s_i X_{i,1}\right\|_{p^*\wedge \mathrm{Log}\, n}.$$

In the case when $X_{i,j}$ are Weibulls with $r \in [1,2]$ or random signs:

$$\mathbb{E}\left\|(X_{i,j})_{i\le m,j\le n}\right\|_{\ell_p^n\to\ell_q^m} \sim m^{1/q}\sup_{t\in B_p^n}\left\|\sum_{j=1}^n t_jX_{1,j}\right\|_{q\wedge \mathrm{Log}\, m} + n^{1/p^*}\sup_{s\in B_{q^*}^m}\left\|\sum_{i=1}^m s_iX_{i,1}\right\|_{p^*\wedge \mathrm{Log}\, n}.$$

Is it true for other random variables?

In the case when $X_{i,j}$ are Weibulls with $r \in [1,2]$ or random signs:

$$\mathbb{E}\Big\|(X_{i,j})_{i\le m, j\le n}\Big\|_{\ell_p^n\to\ell_q^m} \sim m^{1/q} \sup_{t\in B_p^n}\Big\|\sum_{j=1}^{n} t_j X_{1,j}\Big\|_{q\wedge \operatorname{Log} m} + n^{1/p^*} \sup_{s\in B_{q^*}^m}\Big\|\sum_{i=1}^{m} s_i X_{i,1}\Big\|_{p^*\wedge \operatorname{Log} n}.$$

Is it true for other random variables?

Assume that $X_{i,j}$ satisfy

$$\|X_{i,j}\|_{2\rho} \le \alpha \|X_{i,j}\|_{\rho} \quad \text{for all } \rho \ge 1. \qquad (\alpha\text{-reg})$$

In the case when $X_{i,j}$ are Weibulls with $r \in [1,2]$ or random signs:

$$\mathbb{E}\Big\|(X_{i,j})_{i\le m, j\le n}\Big\|_{\ell_p^n\to\ell_q^m}$$
$$\sim m^{1/q}\sup_{t\in B_p^n}\Big\|\sum_{j=1}^n t_j X_{1,j}\Big\|_{q\wedge \operatorname{Log} m} + n^{1/p^*}\sup_{s\in B_{q^*}^m}\Big\|\sum_{i=1}^m s_i X_{i,1}\Big\|_{p^*\wedge \operatorname{Log} n}.$$

Is it true for other random variables?

Assume that $X_{i,j}$ satisfy

$$\|X_{i,j}\|_{2\rho} \le \alpha\|X_{i,j}\|_\rho \quad \text{for all } \rho\ge 1. \qquad (\alpha\text{-reg})$$

Then

$$\mathbb{P}\Big(\sup_{t\in T}\Big|\sum_{i=1}^n t_i X_i\Big| \ge C_1(\alpha)\Big(u+\mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^n t_i X_i\Big|\Big)\Big)$$
$$\le C_2(\alpha)\sup_{t\in T}\mathbb{P}\Big(\Big|\sum_{i=1}^n t_i X_i\Big|\ge u\Big),$$

In the case when $X_{i,j}$ are Weibulls with $r \in [1,2]$ or random signs:

$$\mathbb{E}\Big\|(X_{i,j})_{i\le m, j\le n}\Big\|_{\ell_p^n\to\ell_q^m}$$
$$\sim m^{1/q} \sup_{t\in B_p^n}\Big\|\sum_{j=1}^n t_j X_{1,j}\Big\|_{q\wedge \operatorname{Log} m} + n^{1/p^*} \sup_{s\in B_{q^*}^m}\Big\|\sum_{i=1}^m s_i X_{i,1}\Big\|_{p^*\wedge \operatorname{Log} n}.$$

Is it true for other random variables?

Assume that $X_{i,j}$ satisfy

$$\|X_{i,j}\|_{2\rho} \le \alpha \|X_{i,j}\|_{\rho} \quad \text{for all } \rho \ge 1. \qquad (\alpha\text{-reg})$$

Then

$$\Big\|\sum_{i,j} u_{i,j}X_{i,j}\Big\|_p \lesssim_\alpha \left(\frac{p}{q}\right)^\beta \Big\|\sum_{i,j} u_{i,j}X_{i,j}\Big\|_q \qquad \text{for every } p\ge q\ge 1$$

with $\beta = \frac{1}{2}\vee \log_2\alpha$.

In the case when $X_{i,j}$ are Weibulls with $r \in [1,2]$ or random signs:

$$\mathbb{E}\left\|(X_{i,j})_{i\le m, j\le n}\right\|_{\ell_p^n\to\ell_q^m}$$
$$\sim m^{1/q} \sup_{t\in B_p^n}\left\|\sum_{j=1}^n t_j X_{1,j}\right\|_{q\wedge \operatorname{Log} m} + n^{1/p^*} \sup_{s\in B_{q^*}^m}\left\|\sum_{i=1}^m s_i X_{i,1}\right\|_{p^*\wedge \operatorname{Log} n}.$$

Is it true for other random variables?

Assume that $X_{i,j}$ satisfy

$$\|X_{i,j}\|_{2\rho} \le \alpha \|X_{i,j}\|_\rho \quad \text{for all } \rho \ge 1. \qquad (\alpha\text{-reg})$$

Then

$$\left\|\sum_{i,j} u_{i,j} X_{i,j}\right\|_p \lesssim_\alpha \left(\frac{p}{q}\right)^\beta \left\|\sum_{i,j} u_{i,j} X_{i,j}\right\|_q \qquad \text{for every } p \ge q \ge 1$$

with $\beta = \frac{1}{2} \vee \log_2 \alpha$. In particular $\sum u_{i,j} X_{i,j}$ are $\psi_{1/\beta}$.

Proposition (Latała–Strzelecka 2023+)

If

$$\|X_{i,j}\|_{2\rho} \le \alpha \|X_{i,j}\|_\rho \quad \textit{for all } \rho \ge 1, \qquad (\alpha\text{-reg})$$

then

$$\mathbb{E}\Big\|(X_{i,j})_{i\le m, j\le n}\Big\|_{\ell_p^n \to \ell_q^m}$$

$$\gtrsim_\alpha m^{1/q} \sup_{t\in B_p^n} \Big\|\sum_{j=1}^n t_j X_{1,j}\Big\|_{q\wedge \mathrm{Log}\, m} + n^{1/p^*} \sup_{s\in B_{q^*}^m} \Big\|\sum_{i=1}^m s_i X_{i,1}\Big\|_{p^*\wedge \mathrm{Log}\, n}$$

Conjecture

If

$$\|X_{i,j}\|_{2\rho} \le \alpha \|X_{i,j}\|_\rho \quad \textit{for all } \rho \ge 1, \tag{α-reg}$$

then

$$\mathbb{E}\Big\|(X_{i,j})_{i\le m, j\le n}\Big\|_{\ell_p^n \to \ell_q^m} \tag{$\star$}$$

$$\sim_\alpha m^{1/q} \sup_{t\in B_p^n} \Big\|\sum_{j=1}^n t_j X_{1,j}\Big\|_{q\wedge \mathrm{Log}\, m} + n^{1/p^*} \sup_{s\in B_{q^*}^m} \Big\|\sum_{i=1}^m s_i X_{i,1}\Big\|_{p^*\wedge \mathrm{Log}\, n}$$

Conjecture

If

$$\|X_{i,j}\|_{2\rho} \le \alpha \|X_{i,j}\|_\rho \quad \textit{for all } \rho \ge 1, \qquad (\alpha\text{-reg})$$

then

$$\mathbb{E}\Big\|(X_{i,j})_{i\le m, j\le n}\Big\|_{\ell_p^n \to \ell_q^m} \qquad (\star)$$

$$\sim_\alpha m^{1/q} \sup_{t\in B_p^n} \Big\|\sum_{j=1}^n t_j X_{1,j}\Big\|_{q\wedge \operatorname{Log} m} + n^{1/p^*} \sup_{s\in B_{q^*}^m} \Big\|\sum_{i=1}^m s_i X_{i,1}\Big\|_{p^*\wedge \operatorname{Log} n}$$

Theorem (Latała–S., 2023+)

In the square case $(m = n)$ $(\star)$ *is equivalent to*

$$\mathbb{E}\|(X_{i,j})_{i,j=1}^n\|_{\ell_p^n\to\ell_q^n} \sim_\alpha \begin{cases} n^{1/q+1/p^*-1/2}\|X_{1,1}\|_2, & p^*, q \le 2, \\ n^{1/(p^*\wedge q)}\|X_{1,1}\|_{p^*\wedge q\wedge \operatorname{Log} n}, & p^* \vee q \ge 2 \end{cases}$$

Conjecture

If

$$\|X_{i,j}\|_{2\rho} \le \alpha \|X_{i,j}\|_\rho \quad \textit{for all } \rho \ge 1, \qquad (\alpha\text{-reg})$$

then

$$\mathbb{E}\Big\|(X_{i,j})_{i\le m, j\le n}\Big\|_{\ell_p^n \to \ell_q^m} \qquad (\star)$$

$$\sim_\alpha m^{1/q} \sup_{t\in B_p^n} \Big\|\sum_{j=1}^n t_j X_{1,j}\Big\|_{q\wedge \mathrm{Log}\, m} + n^{1/p^*} \sup_{s\in B_{q^*}^m} \Big\|\sum_{i=1}^m s_i X_{i,1}\Big\|_{p^*\wedge \mathrm{Log}\, n}$$

Theorem (Latała–S., 2023+)

In the square case ($m = n$) ($\star$) *is equivalent to*

$$\mathbb{E}\|(X_{i,j})_{i,j=1}^n\|_{\ell_p^n\to\ell_q^n} \sim_\alpha \begin{cases} n^{1/q+1/p^*-1/2}\|X_{1,1}\|_2, & p^*, q \le 2, \\ n^{1/(p^*\wedge q)}\|X_{1,1}\|_{p^*\wedge q\wedge \mathrm{Log}\, n}, & p^* \vee q \ge 2 \end{cases}$$

and it holds.

Theorem (Latała–S., 2023+)

Assumption (α-reg) *implies* ($\star$) *provided that one of the following conditions holds:*

(i) $p^*, q \le 2$,

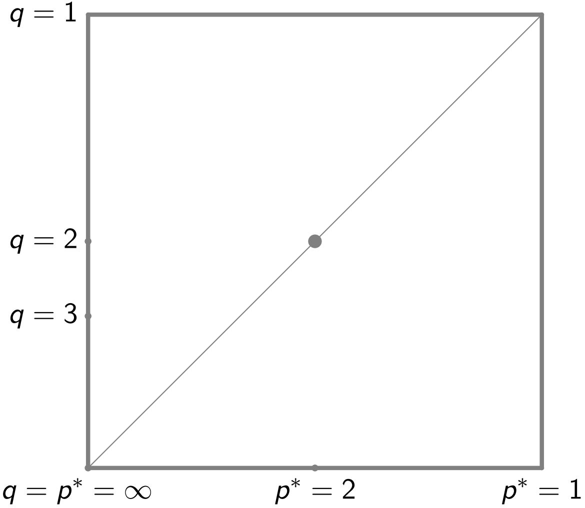
(ii) $p^*, q \ge 3$,

(iii) $p^* \le 3 \le C(\alpha) \le q$ *or* $q \le 3 \le C(\alpha) \le p^*$,

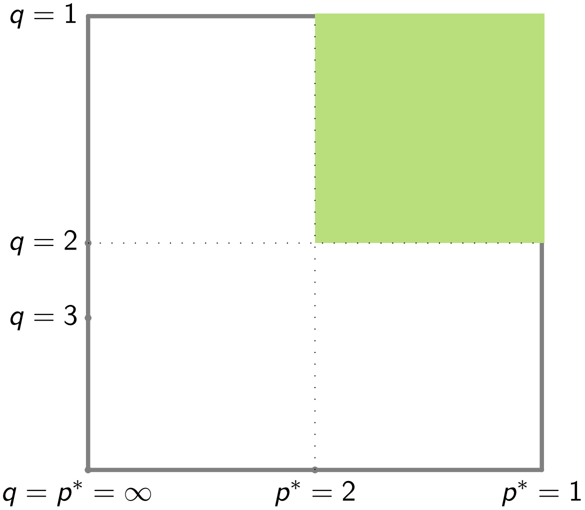
(iv) $p^* \ge \operatorname{Log} n$ *or* $q \ge \operatorname{Log} m$,

(v) $n = m$,

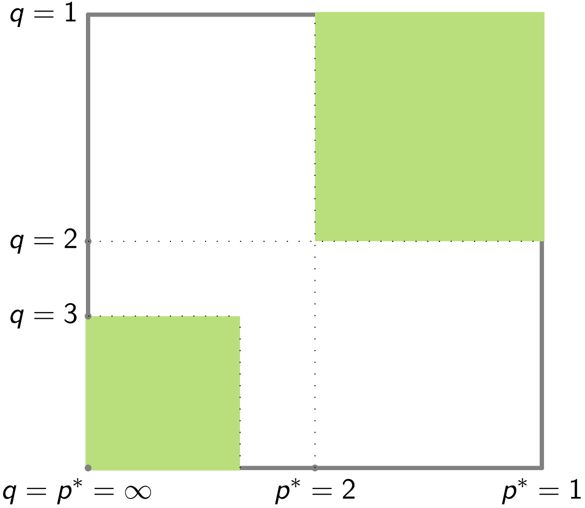
(vi) $X_{i,j}$ *are* $(\sigma\|X_{i,j}\|_2)$*-subexponential (in this case the constant in the upper bound depends also on* σ*).*



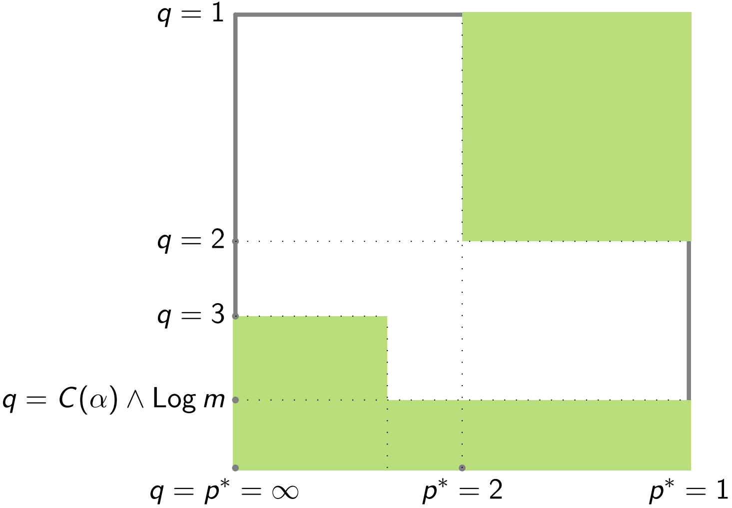
Duality $(p^*, q, n, m) \longleftrightarrow (q, p^*, m, n)$.



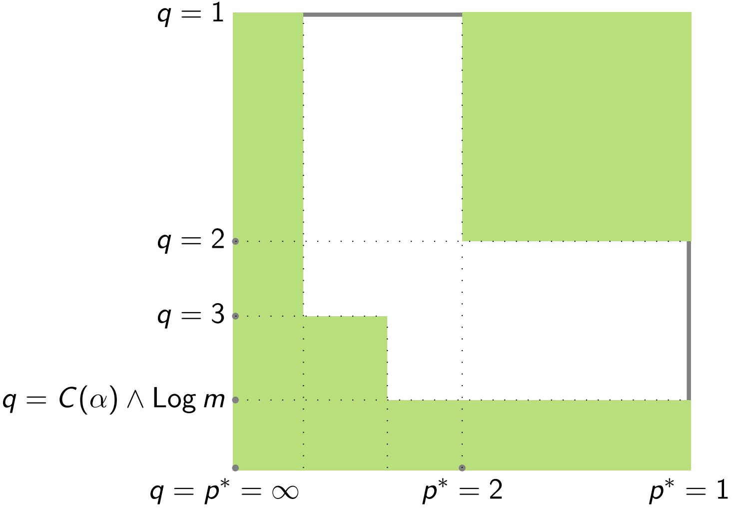
$p^*, q \leq 2$



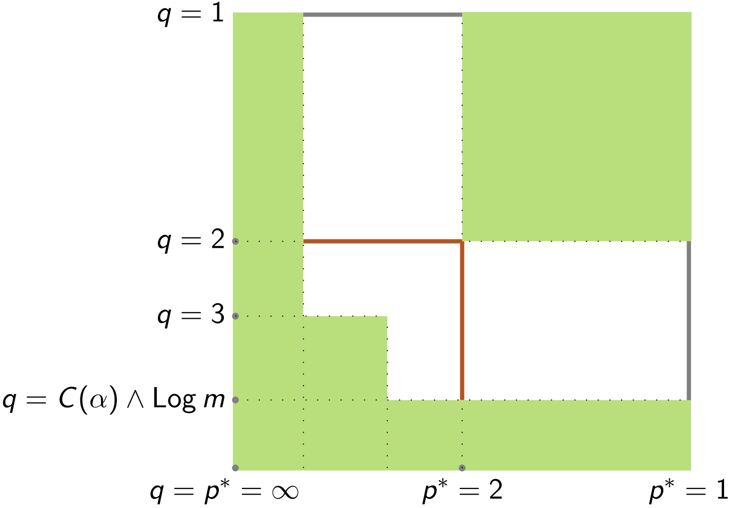
$p^*, q \geq 3$

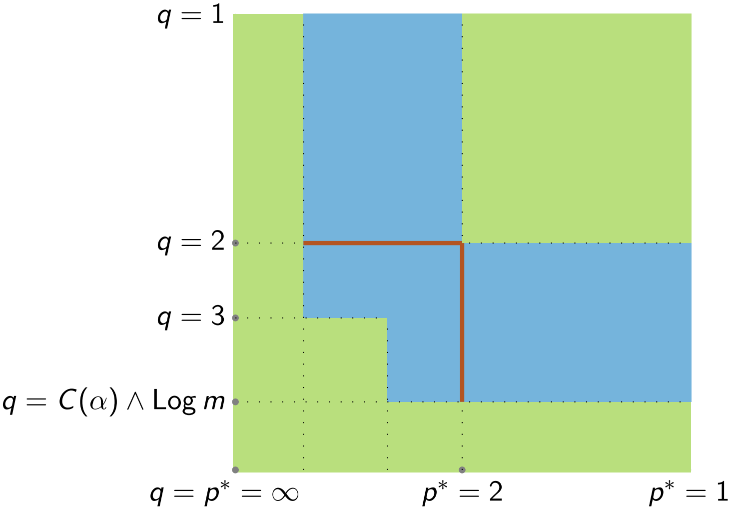


$$p^* \leq 3 \leq C(\alpha) \leq q$$

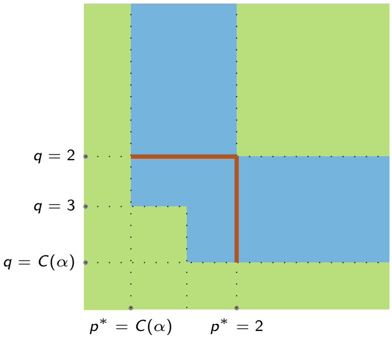


$p^* \leq 3 \leq C(\alpha) \leq q$ or $q \leq 3 \leq C(\alpha) \leq p^*$

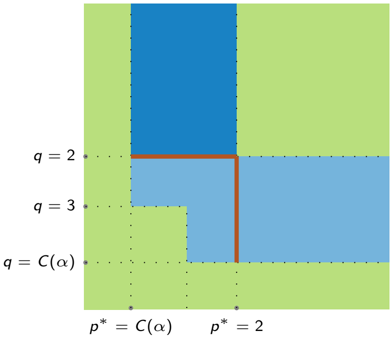

$q = 1$
$q = 2$
$q = 3$
$q = C(\alpha) \wedge \operatorname{Log} m$
$q = p^* = \infty$
$p^* = 2$
$p^* = 1$



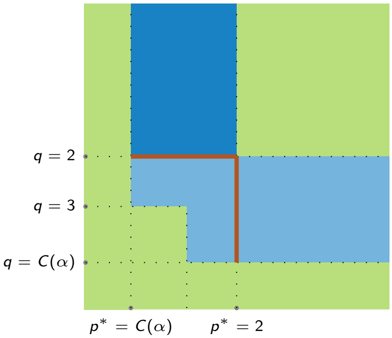
$q = 1$
$q = 2$
$q = 3$
$q = C(\alpha) \wedge \operatorname{Log} m$
$q = p^* = \infty$
$p^* = 2$
$p^* = 1$



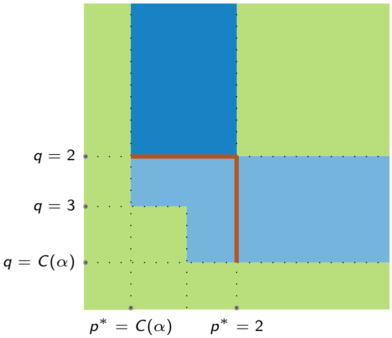
$q = 2$
$q = 3$
$q = C(\alpha)$
$p^* = C(\alpha)$
$p^* = 2$



$q = 2$
$q = 3$
$q = C(\alpha)$
$p^* = C(\alpha)$
$p^* = 2$

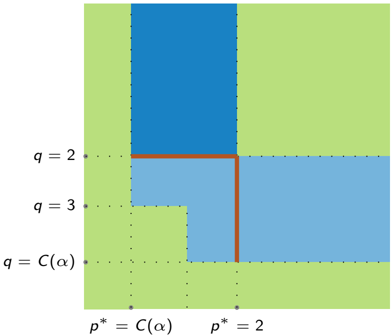


$$m^{1/q}\sup_{t\in B_p^n}\Big\|\sum_{j=1}^n t_j X_{1,j}\Big\|_q$$
$$+n^{1/p^*}\sup_{s\in B_{q^*}^m}\Big\|\sum_{i=1}^m s_i X_{i,1}\Big\|_{p^*}$$
$$\sim_\alpha m^{1/q}\sup_{t\in B_p^n}\|t\|_2$$
$$+n^{1/p^*}\sup_{s\in B_{q^*}^m}\|s\|_2$$
$$= m^{1/q} + n^{1/p^*} m^{1/2-1/q^*}$$



$$
\begin{aligned}
&m^{1/q} \sup_{t\in B_p^n} \Big\| \sum_{j=1}^n t_j X_{1,j} \Big\|_q \\
&\qquad + n^{1/p^*} \sup_{s\in B_{q^*}^m} \Big\| \sum_{i=1}^m s_i X_{i,1} \Big\|_{p^*} \\
&\sim_\alpha m^{1/q} \sup_{t\in B_p^n} \|t\|_2 \\
&\qquad + n^{1/p^*} \sup_{s\in B_{q^*}^m} \|s\|_2 \\
&= m^{1/q} + n^{1/p^*} m^{1/2-1/q^*}
\end{aligned}
$$

$$
\mathbb{E}\|X\|_{\ell_p^n\to\ell_q^m} \le \mathbb{E}\|X\|_{\ell_p^n\to\ell_2^m} \|\operatorname{Id}\|_{\ell_2^m\to\ell_q^m} \sim_\alpha (m^{1/2} + n^{1/p^*}) m^{1/q-1/2}.
$$



$$m^{1/q}\sup_{t\in B_p^n}\Big\|\sum_{j=1}^n t_jX_{1,j}\Big\|_q + n^{1/p^*}\sup_{s\in B_{q^*}^m}\Big\|\sum_{i=1}^m s_iX_{i,1}\Big\|_{p^*}$$

$$\sim_\alpha m^{1/q}\sup_{t\in B_p^n}\|t\|_2 + n^{1/p^*}\sup_{s\in B_{q^*}^m}\|s\|_2$$

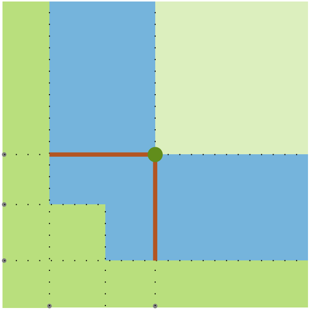
$$= m^{1/q} + n^{1/p^*}m^{1/2-1/q^*}$$

$$\mathbb{E}\|X\|_{\ell_p^n\to\ell_q^m} \le \mathbb{E}\|X\|_{\ell_p^n\to\ell_2^m}\|\operatorname{Id}\|_{\ell_2^m\to\ell_q^m} \sim_\alpha (m^{1/2}+n^{1/p^*})m^{1/q-1/2}.$$

Moreover, for σ-subexponential $X_{i,j}$'s

$$\mathbb{E}\|X\|_{\ell_p^n\to\ell_q^m} \sim_{\alpha,\sigma} m^{1/q}\sup_{t\in B_p^n}\|t\|_2 + n^{1/p^*}\sup_{s\in B_{q^*}^m}\|s\|_2$$

in blue and orange/red regions.

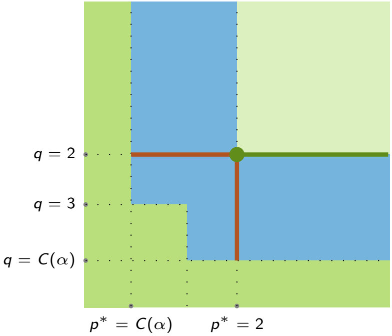


$$
\begin{aligned}
&m^{1/q} \sup_{t\in B_p^n} \Big\| \sum_{j=1}^n t_j X_{1,j} \Big\|_q \\
&\qquad + n^{1/p^*} \sup_{s\in B_{q^*}^m} \Big\| \sum_{i=1}^m s_i X_{i,1} \Big\|_{p^*} \\
&\sim_\alpha m^{1/q} \sup_{t\in B_p^n} \|t\|_2 \\
&\qquad + n^{1/p^*} \sup_{s\in B_{q^*}^m} \|s\|_2 \\
&= m^{1/q} + n^{1/p^*} m^{1/2-1/q^*}
\end{aligned}
$$

Theorem (Latała, 2005)

$$
\mathbb{E}\|X\colon \ell_2^n \to \ell_2^m\|
$$

$$
\lesssim \max_{j\le n}\Big(\sum_{i=1}^m \mathbb{E}X_{ij}^2\Big)^{1/2} + \max_{i\le m}\Big(\sum_{j=1}^n \mathbb{E}X_{ij}^2\Big)^{1/2} + \Big(\sum_{i=1}^m\sum_{j=1}^n \mathbb{E}X_{ij}^4\Big)^{1/4}.
$$



$$m^{1/q}\sup_{t\in B_p^n}\Big\|\sum_{j=1}^n t_j X_{1,j}\Big\|_q + n^{1/p^*}\sup_{s\in B_{q^*}^m}\Big\|\sum_{i=1}^m s_i X_{i,1}\Big\|_{p^*}$$

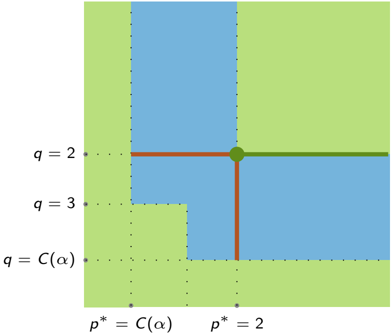
$$\sim_\alpha m^{1/q}\sup_{t\in B_p^n}\|t\|_2 + n^{1/p^*}\sup_{s\in B_{q^*}^m}\|s\|_2$$

$$= m^{1/q} n^{1/2-1/p} + n^{1/p^*}$$

Theorem (Latała, 2005)

$$\mathbb{E}\|X\colon \ell_2^n \to \ell_2^m\|$$

$$\lesssim \max_{j\le n}\Big(\sum_{i=1}^m \mathbb{E}X_{ij}^2\Big)^{1/2} + \max_{i\le m}\Big(\sum_{j=1}^n \mathbb{E}X_{ij}^2\Big)^{1/2} + \Big(\sum_{i=1}^m\sum_{j=1}^n \mathbb{E}X_{ij}^4\Big)^{1/4}.$$



$$m^{1/q} \sup_{t\in B_p^n} \Big\| \sum_{j=1}^n t_j X_{1,j} \Big\|_q + n^{1/p^*} \sup_{s\in B_{q^*}^m} \Big\| \sum_{i=1}^m s_i X_{i,1} \Big\|_{p^*}$$

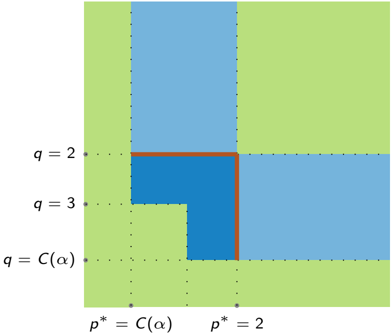
$$\sim_\alpha m^{1/q} \sup_{t\in B_p^n} \|t\|_2 + n^{1/p^*} \sup_{s\in B_{q^*}^m} \|s\|_2$$

$$= m^{1/q} n^{1/2-1/p} + n^{1/p^*} m^{1/2-1/q^*}$$

Theorem (Latała, 2005)

$$\mathbb{E}\|X\colon \ell_2^n \to \ell_2^m\|$$

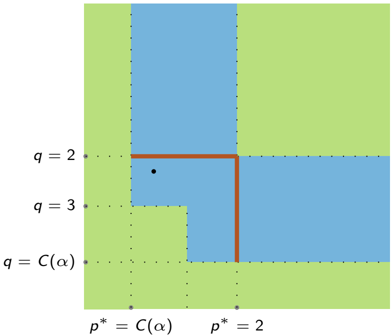
$$\lesssim \max_{j\le n}\Big(\sum_{i=1}^m \mathbb{E}X_{ij}^2\Big)^{1/2} + \max_{i\le m}\Big(\sum_{j=1}^n \mathbb{E}X_{ij}^2\Big)^{1/2} + \Big(\sum_{i=1}^m \sum_{j=1}^n \mathbb{E}X_{ij}^4\Big)^{1/4}.$$



$$m^{1/q}\sup_{t\in B_p^n}\Big\|\sum_{j=1}^n t_j X_{1,j}\Big\|_q + n^{1/p^*}\sup_{s\in B_{q^*}^m}\Big\|\sum_{i=1}^m s_i X_{i,1}\Big\|_{p^*}$$

$$\sim_\alpha m^{1/q}\sup_{t\in B_p^n}\|t\|_2 + n^{1/p^*}\sup_{s\in B_{q^*}^m}\|s\|_2$$

$$= m^{1/q} + n^{1/p^*} \sim n^{1/p^*} \vee m^{1/q}$$

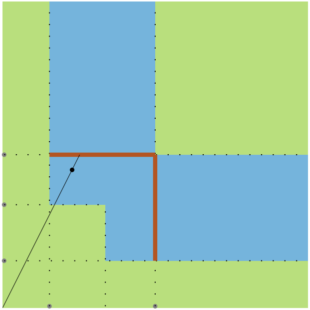


$$m^{1/q} \sup_{t\in B_p^n} \left\| \sum_{j=1}^n t_j X_{1,j} \right\|_q + n^{1/p^*} \sup_{s\in B_{q^*}^m} \left\| \sum_{i=1}^m s_i X_{i,1} \right\|_{p^*}$$

$$\sim_\alpha m^{1/q} \sup_{t\in B_p^n} \|t\|_2 + n^{1/p^*} \sup_{s\in B_{q^*}^m} \|s\|_2$$

$$= m^{1/q} + n^{1/p^*} \sim n^{1/p^*} \vee m^{1/q}$$

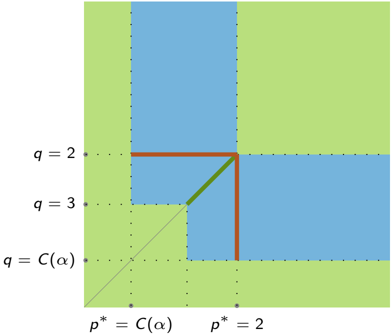
$$\lambda q = p^*, \lambda \in [1, \tfrac{C(\alpha)}{2}].$$



$$
\begin{aligned}
&m^{1/q}\sup_{t\in B_p^n}\Big\|\sum_{j=1}^n t_jX_{1,j}\Big\|_q \\
&\qquad + n^{1/p^*}\sup_{s\in B_{q^*}^m}\Big\|\sum_{i=1}^m s_iX_{i,1}\Big\|_{p^*} \\
&\sim_\alpha m^{1/q}\sup_{t\in B_p^n}\|t\|_2 \\
&\qquad + n^{1/p^*}\sup_{s\in B_{q^*}^m}\|s\|_2 \\
&= m^{1/q}+n^{1/p^*}\sim n^{1/p^*}\vee m^{1/q}
\end{aligned}
$$

$\lambda q = p^*, \lambda\in[1,\frac{C(\alpha)}{2}]$. If $\frac{1}{q}=\frac{\theta}{2}+\frac{1-\theta}{3}$, then $\frac{1}{p^*}=\frac{1}{q\lambda}=\frac{\theta}{2\lambda}+\frac{1-\theta}{3\lambda}$. Thus the Riesz-Thorin interpolation theorem implies

$$
\begin{aligned}
\mathbb{E}\|X\|_{\ell_p^n\to\ell_q^m} &\le \mathbb{E}\|X\|^{\theta}_{\ell^n_{(2\lambda)^*}\to\ell_2^m}\|X\|^{1-\theta}_{\ell^n_{(3\lambda)^*}\to\ell_3^m} \\
&\le (\mathbb{E}\|X\|_{\ell^n_{(2\lambda)^*}\to\ell_2^m})^{\theta}(\mathbb{E}\|X\|_{\ell^n_{(3\lambda)^*}\to\ell_3^m})^{1-\theta} \\
&\sim_\alpha (n^{1/(2\lambda)}\vee m^{1/2})^{\theta}(n^{1/(3\lambda)}\vee m^{1/3})^{\theta} = (n^{1/\lambda}\vee m)^{\frac{\theta}{2}+\frac{1-\theta}{3}} \\
&= (n^{1/\lambda}\vee m)^{1/q} = n^{1/p^*}\vee m^{1/q}.
\end{aligned}
$$



$$\begin{aligned} &m^{1/q} \sup_{t \in B_p^n} \Big\| \sum_{j=1}^n t_j X_{1,j} \Big\|_q \\ &\qquad + n^{1/p^*} \sup_{s \in B_{q^*}^m} \Big\| \sum_{i=1}^m s_i X_{i,1} \Big\|_{p^*} \\ &\sim_\alpha m^{1/q} \sup_{t \in B_p^n} \|t\|_2 \\ &\qquad + n^{1/p^*} \sup_{s \in B_{q^*}^m} \|s\|_2 \\ &= m^{1/q} + n^{1/p^*} \sim n^{1/p^*} \vee m^{1/q} \end{aligned}$$

$\lambda q = p^*, \lambda \in [1, \frac{C(\alpha)}{2}]$. If $\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{3}$, then $\frac{1}{p^*} = \frac{1}{q\lambda} = \frac{\theta}{2\lambda} + \frac{1-\theta}{3\lambda}$. Thus the Riesz-Thorin interpolation theorem implies

$$\begin{aligned} \mathbb{E}\|X\|_{\ell_p^n \to \ell_q^m} &\le \mathbb{E}\|X\|^{\theta}_{\ell^n_{(2\lambda)^*} \to \ell_2^m} \|X\|^{1-\theta}_{\ell^n_{(3\lambda)^*} \to \ell_3^m} \\ &\le (\mathbb{E}\|X\|_{\ell^n_{(2\lambda)^*} \to \ell_2^m})^{\theta} (\mathbb{E}\|X\|_{\ell^n_{(3\lambda)^*} \to \ell_3^m})^{1-\theta} \\ &\sim_\alpha (n^{1/(2\lambda)} \vee m^{1/2})^{\theta} (n^{1/(3\lambda)} \vee m^{1/3})^{\theta} = (n^{1/\lambda} \vee m)^{\frac{\theta}{2} + \frac{1-\theta}{3}} \\ &= (n^{1/\lambda} \vee m)^{1/q} = n^{1/p^*} \vee m^{1/q}. \end{aligned}$$

Theorem (Latała–S., 2023+)

If $m = n$, then

$$\mathbb{E}\big\|(X_{i,j})_{i,j=1}^n\big\|_{\ell_p^n\to\ell_q^n} \sim_\alpha \begin{cases} n^{1/q+1/p^*-1/2}\|X_{1,1}\|_2, & p^*, q \le 2, \\ n^{1/(p^*\wedge q)}\|X_{1,1}\|_{p^*\wedge q\wedge \operatorname{Log} n}, & p^* \vee q \ge 2. \end{cases}$$

Theorem (Latała–S., 2023+)

If $m = n$, then

$$\mathbb{E}\big\|(X_{i,j})_{i,j=1}^n\big\|_{\ell_p^n \to \ell_q^n} \sim_\alpha \begin{cases} n^{1/q+1/p^*-1/2}\|X_{1,1}\|_2, & p^*, q \le 2, \\ n^{1/(p^*\wedge q)}\|X_{1,1}\|_{p^*\wedge q \wedge \operatorname{Log} n}, & p^* \vee q \ge 2. \end{cases}$$

Proof. It suffices to consider $q \ge p^* \ge 2$.

Theorem (Latała–S., 2023+)

If $m = n$, then

$$\mathbb{E}\left\|(X_{i,j})_{i,j=1}^n\right\|_{\ell_p^n\to\ell_q^n} \sim_\alpha \begin{cases} n^{1/q+1/p^*-1/2}\|X_{1,1}\|_2, & p^*, q \le 2, \\ n^{1/(p^*\wedge q)}\|X_{1,1}\|_{p^*\wedge q\wedge \operatorname{Log} n}, & p^* \vee q \ge 2. \end{cases}$$

Proof. It suffices to consider $q \ge p^* \ge 2$. LHS is non-increasing with q, so it suffices to consider $3 \ge q = p^* \ge 2$.

Theorem (Latała–S., 2023+)

If $m = n$, then

$$\mathbb{E}\big\|(X_{i,j})_{i,j=1}^n\big\|_{\ell_p^n\to\ell_q^n} \sim_\alpha \begin{cases} n^{1/q+1/p^*-1/2}\|X_{1,1}\|_2, & p^*, q \le 2, \\ n^{1/(p^*\wedge q)}\|X_{1,1}\|_{p^*\wedge q\wedge \operatorname{Log} n}, & p^* \vee q \ge 2. \end{cases}$$

Proof. It suffices to consider $q \ge p^* \ge 2$. LHS is non-increasing with q, so it suffices to consider $3 \ge q = p^* \ge 2$.
OK for $q \in \{2, 3\}$, so we may interpolate along the diagonal.

Idea of a proof in the rectangular case

$$T_1 = B_p^n \cap k^{-1/p} B_\infty^n, \quad T_2 = \{t \in B_p^n \colon |\mathrm{supp}(t)| \le k\}.$$

Then $B_p^n \subset T_1 + T_2$.

Idea of a proof in the rectangular case

$$T_1 = B_p^n \cap k^{-1/p} B_\infty^n, \qquad T_2 = \{t \in B_p^n \colon |\mathrm{supp}(t)| \le k\}.$$

Then $B_p^n \subset T_1 + T_2$.

Lemma

Let $\beta = \log_2(2\alpha)$. Then $\mathbb{E}\max_{i\le m, j\le n}|X_{ij}| \lesssim \mathrm{Log}^\beta(nm)$, so

$$\begin{aligned}
\mathbb{E}\sup_{s\in S, t\in T}\sum_{i\le m, j\le n} X_{i,j}s_i t_j &\lesssim \mathrm{Log}^\beta(nm)\mathbb{E}\sup_{s\in S, t\in T}\sum_{i\le m, j\le n}\varepsilon_{i,j}s_i t_j \\
&\lesssim \mathrm{Log}^\beta(nm)\mathbb{E}\sup_{s\in S, t\in T}\sum_{i\le m, j\le n} g_{i,j}s_i t_j. \\
&\sim \mathrm{Log}^\beta(nm)\sup_{s\in S}\|s\|_2\,\mathbb{E}\sup_{t\in T}\sum_{j=1}^n g_j t_j \\
&\quad + \mathrm{Log}^\beta(nm)\sup_{t\in T}\|t\|_2\,\mathbb{E}\sup_{s\in S}\sum_{i=1}^m g_i s_i.
\end{aligned}$$

Idea of a proof in the rectangular case

$$T_1 = B_p^n \cap k^{-1/p} B_\infty^n, \quad T_2 = \{t \in B_p^n \colon |\mathrm{supp}(t)| \le k\}.$$

Then $B_p^n \subset T_1 + T_2$.

Lemma

Let $\beta = \frac{1}{2} \vee \log_2 \alpha$.
If $S \subset B_{q^*}^m \cap aB_\infty^m$ and $T \subset \{t \in B_p^n \colon |\mathrm{supp}(t)| \le k\} \cap bB_\infty^n$, then

$$\mathbb{E} \sup_{s \in S, t \in T} \sum_{i \le m, j \le n} X_{i,j} s_i t_j$$
$$\lesssim_\alpha m^{1/q} \sup_{t \in T} \Big\| \sum_{j=1}^n X_{1,j} t_j \Big\|_q + (n \wedge (k \operatorname{Log} n))^\beta ab$$
$$+ (n \wedge (k \operatorname{Log} n))^{1/(2 \wedge p^*)} \begin{cases} a^{(2-q^*)/2}, & q > 2 \\ m^{1/q-1/2}, & q \le 2 \end{cases}.$$

Proof of the lemma

W.l.o.g. $X_{i,j}$'s are symmetric and $k \operatorname{Log} n \leq n$.

$$T_0 \subset T \subset \{t \colon |\mathrm{supp} t| \leq k\}.$$

T_0 – a $\frac{1}{2}$-net in T w.r.t. ℓ_p-metric.

Proof of the lemma

W.l.o.g. $X_{i,j}$'s are symmetric and $k \operatorname{Log} n \leq n$.

$$T_0 \subset T \subset \{t \colon |\mathrm{supp} t| \leq k\}.$$

T_0 – a $\frac{1}{2}$-net in T w.r.t. ℓ_p-metric. $|T_0| \leq \binom{n}{k} 5^k \leq (5n)^k = e^d$, where $d \sim k \ln n$.

Proof of the lemma

W.l.o.g. $X_{i,j}$'s are symmetric and $k \operatorname{Log} n \leq n$.

$$T_0 \subset T \subset \{t \colon |\mathrm{supp} t| \leq k\}.$$

T_0 – a $\frac{1}{2}$-net in T w.r.t. ℓ_p-metric. $|T_0| \leq \binom{n}{k} 5^k \leq (5n)^k = e^d$, where $d \sim k \ln n$. Then

$$\begin{aligned}
\mathbb{E} \sup_{s \in S, t \in T} \sum_{i \leq m, j \leq n} X_{i,j} s_i t_j &\leq 2\mathbb{E} \sup_{t \in T_0, s \in S} \sum_{i \leq m, j \leq n} X_{i,j} s_i t_j \\
&\leq 2\Big(\mathbb{E} \sum_{t \in T_0} \sup_{s \in S} \Big| \sum_{i \leq m, j \leq n} X_{i,j} s_i t_j \Big|^d \Big)^{1/d} \\
&\leq 2|T_0|^{1/d} \sup_{t \in T_0} \Big(\mathbb{E} \sup_{s \in S} \Big| \sum_{i \leq m, j \leq n} X_{i,j} s_i t_j \Big|^d \Big)^{1/d} \\
&\leq 2e \sup_{t \in T} \Big(\mathbb{E} \sup_{s \in S} \Big| \sum_{i \leq m, j \leq n} X_{i,j} s_i t_j \Big|^d \Big)^{1/d}.
\end{aligned}$$

Proof of the lemma

W.l.o.g. $X_{i,j}$'s are symmetric and $k \operatorname{Log} n \leq n$.

$$T_0 \subset T \subset \{t \colon |\operatorname{supp} t| \leq k\}.$$

T_0 – a $\frac{1}{2}$-net in T w.r.t. ℓ_p-metric. $|T_0| \leq \binom{n}{k} 5^k \leq (5n)^k = e^d$, where $d \sim k \ln n$. Then

$$\begin{aligned}
\mathbb{E} \sup_{s \in S, t \in T} \sum_{i \leq m, j \leq n} X_{i,j} s_i t_j &\leq 2\mathbb{E} \sup_{t \in T_0, s \in S} \sum_{i \leq m, j \leq n} X_{i,j} s_i t_j \\
&\leq 2\Big(\mathbb{E} \sum_{t \in T_0} \sup_{s \in S} \Big| \sum_{i \leq m, j \leq n} X_{i,j} s_i t_j \Big|^d \Big)^{1/d} \\
&\leq 2|T_0|^{1/d} \sup_{t \in T_0} \Big(\mathbb{E} \sup_{s \in S} \Big| \sum_{i \leq m, j \leq n} X_{i,j} s_i t_j \Big|^d \Big)^{1/d} \\
&\leq 2e \sup_{t \in T} \Big(\mathbb{E} \sup_{s \in S} \Big| \sum_{i \leq m, j \leq n} X_{i,j} s_i t_j \Big|^d \Big)^{1/d}.
\end{aligned}$$

Fix $t \in T$.

$$\Big(\mathbb{E}\sup_{s\in S}\Big|\sum_{i\le m, j\le n} X_{i,j} s_i t_j\Big|^d\Big)^{1/d} \lesssim_\alpha \mathbb{E}\sup_{s\in S}\Big|\sum_{i\le m, j\le n} X_{i,j} s_i t_j\Big| + \sup_{s\in S}\Big\|\sum_{i\le m, j\le n} X_{i,j} s_i t_j\Big\|_d.$$

Fix $t \in T$.

$$\Big(\mathbb{E}\sup_{s\in S}\Big|\sum_{i\le m, j\le n} X_{i,j}s_i t_j\Big|^d\Big)^{1/d} \lesssim_\alpha \mathbb{E}\sup_{s\in S}\Big|\sum_{i\le m, j\le n} X_{i,j}s_i t_j\Big| + \sup_{s\in S}\Big\|\sum_{i\le m, j\le n} X_{i,j}s_i t_j\Big\|_d.$$

Fix $t \in T$.

$$\Big(\mathbb{E}\sup_{s\in S}\Big|\sum_{i\le m, j\le n} X_{i,j}s_i t_j\Big|^d\Big)^{1/d} \lesssim_\alpha \mathbb{E}\sup_{s\in S}\Big|\sum_{i\le m, j\le n} X_{i,j}s_i t_j\Big| + \sup_{s\in S}\Big\|\sum_{i\le m, j\le n} X_{i,j}s_i t_j\Big\|_d.$$

Since $S \subset B_{q^*}^m$,

$$\mathbb{E}\sup_{s\in S}\Big|\sum_{i\le m, j\le n} X_{i,j}s_i t_j\Big| \le \mathbb{E}\Big\|\Big(\sum_{j=1}^n X_{i,j}t_j\Big)_{i\le m}\Big\|_q \le \Big(\mathbb{E}\Big\|\Big(\sum_{j=1}^n X_{i,j}t_j\Big)_{i\le m}\Big\|_q^q\Big)^{1/q} = m^{1/q}\Big\|\sum_{j=1}^n X_{1,j}t_j\Big\|_q.$$

Fix $t \in T$.

$$\Big(\mathbb{E}\sup_{s\in S}\Big|\sum_{i\le m, j\le n} X_{i,j}s_i t_j\Big|^d\Big)^{1/d} \lesssim_\alpha \mathbb{E}\sup_{s\in S}\Big|\sum_{i\le m, j\le n} X_{i,j}s_i t_j\Big| + \sup_{s\in S}\Big\|\sum_{i\le m, j\le n} X_{i,j}s_i t_j\Big\|_d.$$

$$\mathbb{E}\sup_{s\in S}\Big|\sum_{i\le m, j\le n} X_{i,j}s_i t_j\Big| \le m^{1/q}\Big\|\sum_{j=1}^{n} X_{1,j}t_j\Big\|_q.$$

Fix $t \in T$.

$$\Big(\mathbb{E}\sup_{s\in S}\Big|\sum_{i\le m, j\le n} X_{i,j} s_i t_j\Big|^d\Big)^{1/d} \lesssim_\alpha \mathbb{E}\sup_{s\in S}\Big|\sum_{i\le m, j\le n} X_{i,j} s_i t_j\Big| + \sup_{s\in S}\Big\|\sum_{i\le m, j\le n} X_{i,j} s_i t_j\Big\|_d.$$

$$\mathbb{E}\sup_{s\in S}\Big|\sum_{i\le m, j\le n} X_{i,j} s_i t_j\Big| \le m^{1/q}\Big\|\sum_{j=1}^{n} X_{1,j} t_j\Big\|_q.$$

$X_{i,j}$'s are $\psi_{1/\beta}$ (w.l.o.g. $\beta \ge 1$), so

$$\sup_{s\in S}\Big\|\sum_{i\le m, j\le n} X_{i,j} s_i t_j\Big\|_d \lesssim_\alpha d^{1/2}\sup_{s\in S}\|s\|_2\|t\|_2 + d^\beta \sup_{s\in S}\|s\|_\infty\|t\|_\infty$$

Fix $t \in T$.

$$\left(\mathbb{E}\sup_{s\in S}\Big|\sum_{i\le m, j\le n} X_{i,j}s_it_j\Big|^d\right)^{1/d} \lesssim_\alpha \mathbb{E}\sup_{s\in S}\Big|\sum_{i\le m, j\le n} X_{i,j}s_it_j\Big| + \sup_{s\in S}\Big\|\sum_{i\le m, j\le n} X_{i,j}s_it_j\Big\|_d.$$

$$\mathbb{E}\sup_{s\in S}\Big|\sum_{i\le m, j\le n} X_{i,j}s_it_j\Big| \le m^{1/q}\Big\|\sum_{j=1}^n X_{1,j}t_j\Big\|_q.$$

$X_{i,j}$'s are $\psi_{1/\beta}$ (w.l.o.g. $\beta \ge 1$), so

$$\sup_{s\in S}\Big\|\sum_{i\le m, j\le n} X_{i,j}s_it_j\Big\|_d \lesssim_\alpha d^{1/2}\sup_{s\in S}\|s\|_2\|t\|_2 + d^\beta\sup_{s\in S}\|s\|_\infty\|t\|_\infty$$

$$\le d^{1/2}\|t\|_p d^{(-1/p+1/2)\vee 0}\sup_{s\in S}\begin{cases}\|s\|_\infty^{(2-q^*)/2}\|s\|_{q^*}^{q^*/2}, & q>2\\ m^{1/q-1/2}\|s\|_{q^*}, & q\le 2\end{cases} + d^\beta ab$$

Fix $t \in T$.

$$\Big(\mathbb{E}\sup_{s\in S}\Big|\sum_{i\le m, j\le n} X_{i,j}s_i t_j\Big|^d\Big)^{1/d} \lesssim_\alpha \mathbb{E}\sup_{s\in S}\Big|\sum_{i\le m, j\le n} X_{i,j}s_i t_j\Big| + \sup_{s\in S}\Big\|\sum_{i\le m, j\le n} X_{i,j}s_i t_j\Big\|_d.$$

$$\mathbb{E}\sup_{s\in S}\Big|\sum_{i\le m, j\le n} X_{i,j}s_i t_j\Big| \le m^{1/q}\Big\|\sum_{j=1}^n X_{1,j}t_j\Big\|_q.$$

$X_{i,j}$'s are $\psi_{1/\beta}$ (w.l.o.g. $\beta \ge 1$), so

$$\sup_{s\in S}\Big\|\sum_{i\le m, j\le n} X_{i,j}s_i t_j\Big\|_d \lesssim_\alpha d^{1/2}\sup_{s\in S}\|s\|_2\|t\|_2 + d^\beta\sup_{s\in S}\|s\|_\infty\|t\|_\infty$$

$$\le d^{1/2}\|t\|_p d^{(-1/p+1/2)\vee 0}\sup_{s\in S}\begin{cases}\|s\|_\infty^{(2-q^*)/2}\|s\|_{q^*}^{q^*/2}, & q>2\\ m^{1/q-1/2}\|s\|_{q^*}, & q\le 2\end{cases} + d^\beta ab$$

$$\le d^{1/(2\wedge p^*)}\begin{cases}a^{(2-q^*)/2}, & q>2\\ m^{1/q-1/2}, & q\le 2\end{cases} + d^\beta ab.$$