Magnitude and intrinsic volumes of convex bodies

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Magnitude: the origin story

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$\Rightarrow$ metric space

(Leinster, 2008)

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(Leinster, 2008)

We end up with something quite different from the Euler characteristic of the underlying topological space of a metric space — an isometric invariant, not a topological one.
We need a new name!

Finite metric spaces

Definition (Leinster, 2008)

Let (A, d) *be a finite metric space.*
Define $Z_A \in \mathbb{R}^{A \times A}$ *by* $Z_A(a, b) = e^{-d(a,b)}$.
The magnitude of A *is*

$$\mathrm{Mag}(A) = \sum_{a,b \in A} Z_A^{-1}(a, b) = \sum_{a \in A} (Z_A^{-1} 1)(a)$$

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The magnitude function of A is $t \mapsto \mathrm{Mag}(tA)$.

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The magnitude function of A is $t \mapsto \mathrm{Mag}(tA)$.

If Z_{tA} is invertible for all $t > 0$, we say A has negative type.
An infinite space has negative type if all its finite subsets do.
L_1 (and hence ℓ_1^n and ℓ_2^n) has negative type.

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Proposition (Leinster–Willerton, 2013)

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Theorem (Leinster, 2019)

If $G = (V, E)$ is a graph with the shortest path metric, then

$$\mathrm{Mag}(tG) = \#V - 2(\#E)q + \sum_{n=2}^{\infty} c_n q^n,$$

for $q = e^{-t}$ with c_n given explicitly in terms of paths of length n in G.

Aside: magnitude homology

The Euler characteristic of a topological space can be expressed in terms of its homology groups:

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For a finite metric space A there is a family of magnitude homology groups $MH_{n,\ell}(A)$ ($n \in \mathbb{Z}_{\geq 0}$, $\ell \in \mathbb{R}_{\geq 0}$) such that

$$\operatorname{Mag}(tA) = \sum_{\ell \geq 0} \sum_{n=0}^{\infty} (-1)^n (\operatorname{rank} MH_{n,\ell}(A)) q^{\ell}.$$

(Hepworth–Willerton 2017, Leinster–Shulman, etc.)

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Magnitude homology is closely related to persistent homology from topological data analysis (Otter, Cho, Govc–Hepworth).

Compact metric spaces: definition of magnitude

Proposition (M., 2013)

Magnitude is lower semicontinuous on the class of finite metric spaces of negative type.
It therefore has a canonical (maximal) l.s.c. extension to the class of compact metric spaces of negative type, given by

$$\mathrm{Mag}(A) := \sup\{\mathrm{Mag}(B) \mid B \subseteq A \textit{ is finite}\} \in [1, \infty].$$

Compact metric spaces: computation of magnitude

Corollary (M., 2013)

Suppose A is of negative type, $A_m \subseteq A$ for each m, and $A_m \xrightarrow{m\to\infty} A$ in the Hausdorff metric. Then $\operatorname{Mag}(A) = \lim_{m\to\infty} \operatorname{Mag}(A_m)$.

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A finite signed measure μ on A is called a weight measure if

$$\int_A e^{-d(a,b)}\, d\mu(a) = 1 \qquad \forall b \in A.$$

μ is analogous to $Z_A^{-1}1$. Usually no such measure exists.

Proposition (M., 2013)

If A is of negative type and possesses a weight measure μ, then $\mathrm{Mag}(A) = \mu(A)$.

Magnitude in Euclidean space

Proposition (M., 2015)

If $A \subseteq \ell_2^n$ is compact, then

$$\mathrm{Mag}(A) = \inf \left\{ \frac{1}{n!\omega_n} \int_{\mathbb{R}^n} f(x)(\mathrm{I} - \Delta)^{(n+1)/2} f(x)\, dx : f \equiv 1 \text{ on } A \right\},$$

where $\omega_n = \mathrm{vol}_n(B_2^n)$.

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Proposition (Barceló–Carbery, 2018)

If n is odd and $A \subseteq \ell_2^n$ is nice enough, the infimum is achieved by the unique $f \in H^{(n+1)/2}(\mathbb{R}^n)$ with $f \equiv 1$ on A and

$$(I - \Delta)^{(n+1)/2} f \equiv 0 \text{ on } \mathbb{R}^n \setminus A.$$

Magnitude in Euclidean space: rough asymptotics

Theorem (Barceló–Carbery, 2018)

If $A \subseteq \ell_2^n$ is compact, then

$$\lim_{t \to 0} \mathrm{Mag}(tA) = 1 \quad \textit{and} \quad \lim_{t \to \infty} \frac{\mathrm{Mag}(tA)}{t^n} = \frac{\mathrm{vol}(A)}{n!\,\omega_n}.$$

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Theorem (M., 2015)

If $A \subseteq \ell_2^n$ is compact, then

$$\lim_{t\to\infty} \frac{\log \mathrm{Mag}(tA)}{\log t} = \lim_{\varepsilon\to 0} \frac{\log N(A,\varepsilon)}{\log(1/\varepsilon)} = \dim_{\mathrm{Mink}}(A).$$

Magnitude in Euclidean space: odd balls

The only convex sets in Euclidean space whose magnitude is known exactly are odd-dimensional Euclidean balls (Barceló–Carbery 2018, Willerton 2020).

- $\mathrm{Mag}(tB_1^1) = \mathrm{Mag}([-t, t]) = 1 + t$
- $\mathrm{Mag}(tB_2^3) = 1 + t + t^2 + \frac{1}{6}t^3$
- $\mathrm{Mag}(tB_2^5) = \frac{24 + 72t + 72t^2 + 35t^3 + 9t^4 + t^5}{8(t+3)} + \frac{1}{120}t^5$
- $\mathrm{Mag}(tB_2^n)$ is a rational function of t with positive integer coefficients (for n odd).

Magnitude in Euclidean space: finer asymptotics

Suppose n is odd and $A, B \subseteq \ell_2^n$ are nice enough.

Theorem (Gimperlein–Goffeng)

$$\operatorname{Mag}(tA) = \frac{1}{n!\omega_n}\Bigg(\operatorname{vol}(A)t^n + \frac{n+1}{2}\operatorname{vol}_{n-1}(\partial A)t^{n-1} + \frac{(n-1)(n+1)^2}{8}\left(\int_{\partial A} H\, dS\right)t^{n-2}\Bigg) + O(t^{n-3})$$

as $t \to \infty$.

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$$\mathrm{Mag}(t(A \cup B)) + \mathrm{Mag}(t(A \cap B)) - \mathrm{Mag}(tA) - \mathrm{Mag}(tB) \xrightarrow{t \to \infty} 0.$$

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These are proved using semiclassical analysis of the differential operators $(t^2 - \Delta)^{(n+1)/2}$, related to proofs of the Atiyah–Singer index theorem.

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- Kubota's formula:

$$V_i(K) = \binom{n}{i} \frac{\omega_n}{\omega_i \omega_{n-i}} \int_{\mathrm{Gr}_{n,i}} \mathrm{vol}_i\big(\pi_P(K)\big) \, d\mu_{n,i}(P),$$

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- Curvature integral formulas: if ∂K is smoooth,

$$V_i(K) = \binom{n}{i} \frac{1}{n\omega_{n-i}} \int_{\partial K} H_{n-i-1}\, d\mathcal{H}_{n-1},$$

where H_j is the j^{th} normalized elementary symmetric function of the principle curvatures on ∂K.

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  where H_j is the j^{th} normalized elementary symmetric function of the principle curvatures on ∂K.

They are normalized so that $V_i(K) = \mathrm{vol}_i(K)$ if $\dim K = i$.

Magnitude and intrinsic volumes

Theorem (Gimperlein–Goffeng)

If n is odd and $K \subseteq \ell_2^n$ is convex with smooth boundary, then

$$\mathrm{Mag}(tK) = \frac{1}{n!\omega_n}\Big(V_n(K)t^n + (n+1)V_{n-1}(K)t^{n-1} + \frac{\pi}{4}(n+1)^2 V_{n-2}(K)t^{n-2}\Big) + O\big(t^{n-3}\big)$$

as $t \to \infty$.

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as $t \to \infty$.

However, the t^{n-3} term is not proportional to an intrinsic volume.

Magnitude and intrinsic volumes

The main result for today's talk:

Theorem (M., 2020)

If $K \subseteq \ell_2^n$ is compact and convex, then

$$\mathrm{Mag}(K) \leq \sum_{i=0}^{n} \frac{\omega_i}{4^i} V_i(K) \leq \sum_{i=0}^{n} \frac{\omega_i}{4^i i!} V_1(K)^i.$$

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Corollary

If $K \subseteq \ell_2^n$ is compact and convex, then

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as $t \to 0$.

Equality holds here if n is odd and $K = B_2^n$ (Willerton, M.).
The second-order term is almost known in this case (Liu).

Magnitude in Hilbert space

Suppose $K \subseteq \ell_2$ is compact and convex.

$V_1(K) := \sup\{V_1(F) \mid F \subseteq K \text{ compact, convex with } \dim F < \infty\}$.

Sets with $V_1(K) < \infty$ are called GB bodies (for Gaussian bounded).

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Corollary

If $K \subseteq \ell_2$ is a GB body, then $\mathrm{Mag}(K) < \infty$ and $\lim_{t \to 0} \mathrm{Mag}(tK) = 1$.

Intrinsic volumes in ℓ_1^n

Definition (Leinster, 2012)

The ℓ_1-intrinsic volumes of an ℓ_1-convex set $K \subseteq \mathbb{R}^n$ are given by

$$V_i'(K) = \sum_{P \in \mathrm{Gr}_{n,i}'} \mathrm{vol}_i\big(\pi_P(K)\big),$$

where $\mathrm{Gr}_{n,i}'$ is the set of i-dimensional coordinate subspaces of $\mathbb{R}^n$.

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Leinster developed a theory of integral geometry in ℓ_1^n that parallels the classical ℓ_2^n theory (Hadwiger, Steiner, Crofton, kinematic formulas).

Magnitude in ℓ_1^n

The upper bound $\mathrm{Mag}(K) \leq \sum_{i=0}^{n} \frac{\omega_i}{4^i} V_i(K)$ for $K \subseteq \ell_2^n$ follows from:

Theorem (Leinster–Meckes, 2017)

If $K \subseteq \ell_1^n$ is compact and ℓ_1-convex (in particular, if K is convex) then

$$\mathrm{Mag}(K) \leq \sum_{i=0}^{n} 2^{-i} V_i'(K).$$

Equality holds if K has nonempty interior.

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- $\mu_{a,b} := \frac{1}{2}(\delta_a + \delta_b + \lambda_{[a,b]})$ is a weight measure for $[a, b]$.

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- $\mu_{a_1,b_1} \otimes \cdots \otimes \mu_{a_n,b_n}$ is a weight measure for the box $\prod_{i=1}^{n}[a_i, b_i] \subseteq \ell_1^n$.
  $\rightsquigarrow$ Theorem holds with $=$ for boxes.

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- Using Groemer's extension theorem, weight measures can be described explicitly for finite unions of boxes.
  $\rightsquigarrow$ Theorem holds with $=$ for finite unions of boxes.
- K can be approximated by a sequence of finite unions of boxes, and magnitude is l.s.c.

From ℓ_1^N to ℓ_2^n

To get from:

Theorem (Leinster, 2017)

If $K \subseteq \ell_1^N$ is compact and convex then $\operatorname{Mag}(K) \leq \sum_{i=0}^{N} \frac{1}{2^i} V_i'(K)$.

to:

Theorem (M., 2020)

If $K \subseteq \ell_2^n$ is compact and convex, then $\operatorname{Mag}(K) \leq \sum_{i=0}^{n} \frac{\omega_i}{4^i} V_i(K)$

we approximate ℓ_2^n by n-dimensional subspaces of ℓ_1^N with $N \to \infty$.

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- We can use probabilistic ideas to describe such subspaces deterministically. (More elementary.)

From ℓ_1^N to ℓ_2^n

- If $Z_1, \ldots, Z_n$ are independent $N(0,1)$ random variables,

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- Combining these, we get explicit embeddings $T_m : \ell_2^n \to \ell_1^{2^m n}$ with $\|T_m(x)\|_1 \approx \sqrt{\frac{2}{\pi}} 2^{mn} \|x\|_2$.

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- Approximate $K \subseteq \ell_2^n$ by $K_m = \sqrt{\frac{\pi}{2}} 2^{-mn} T_m(K) \subseteq \ell_1^{2^m n}$.

  $\mathrm{Mag}(K) \leq \underline{\lim}_{m\to\infty} \mathrm{Mag}(K_m)$ by semicontinuity. $\square$

The end

Thank you!

References:

https://www.maths.ed.ac.uk/~tl/magbib/