On the eigenvalues of Brownian motion on $\mathbb{U}(N)$

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Brownian motion on $\mathbb{U}(N)$

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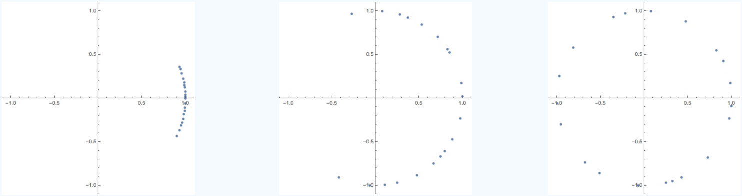
We may define a standard Brownian motion $\{U_t^N\}_{t\geq 0}$ on $\mathbb{U}(N)$ to be a solution to

$$\begin{aligned} dU_t^N &= U_t^N \circ dW_t^N \\ &= U_t^N dW_t^N - \frac{1}{2} U_t^N dt \end{aligned}$$

with $U_0^N = I_N$ and W_t^N a standard B.M. on $\mathfrak{u}(N)$.

The empirical spectral measure



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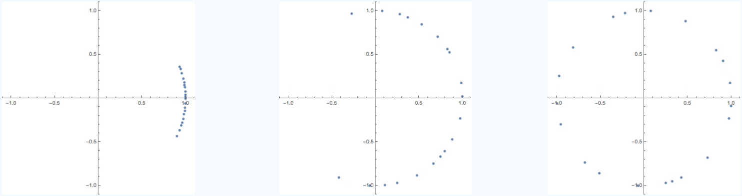


Suppose that M is an $n \times n$ random matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$.

The empirical spectral measure μ of M is the (random) measure

$$\mu := \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k}.$$

The empirical spectral measure

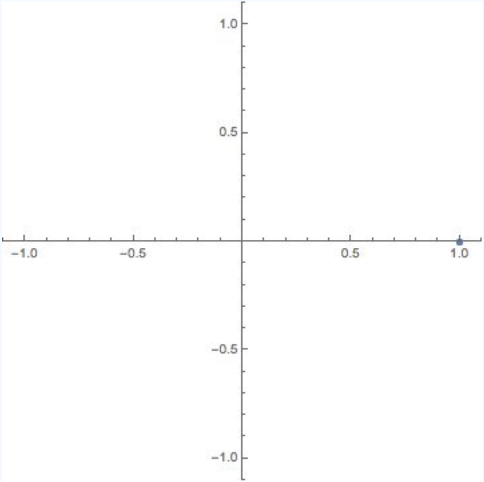
If $\{U_t^N\}_{t\geq 0}$ is a Brownian motion on $\mathbb{U}(N)$, then for each t, U_t^N has N eigenvalues

$$z_{t,1}, \ldots, z_{t,N}$$

on the unit circle, and an associated spectral measure

$$\mu_t^N = \frac{1}{N}\sum_{j=1}^{N} \delta_{z_{t,j}}.$$

The process $\{\mu_t^N\}_{t\geq 0}$



The large N limit

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Theorem (Biane, 1997)

There is a deterministic family of measures $\{\nu_t\}_{t\geq 0}$ on $\mathbb{S}^1$ such that, for each $t \geq 0$, the spectral measure of U_t^N converges weakly almost surely to ν_t:

for all $f \in C(\mathbb{S}^1)$,

$$\lim_{N\to\infty} \int f d\mu_t^N = \int f d\nu_t \qquad a.s.$$

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The measure ν_t represents in some sense the spectral distribution of a "free unitary Brownian motion".

The measures ν_t are characterized in terms of their moments. They have densities (symmetric about 1) on the circle, and are supported on symmetric arcs until time $t = 4$, when their support becomes the whole circle.

Non-asymptotic theory

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L^1-Kantorovich distance:
For Borel probability measures μ and ν on a Polish space $(\mathcal{X}, \rho)$,

$$W_1(\mu, \nu) = \inf \left\{ \int \rho(x, y) d\pi(x, y) : \pi \text{ a coupling of } \mu, \nu \right\}$$

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$$W_1(\mu,\nu) = \inf\left\{\int \rho(x,y)d\pi(x,y) : \pi \text{ a coupling of } \mu,\nu\right\}$$
$$= \sup\left\{\int f d\mu - \int f d\nu : |f|_{Lip} \leq 1\right\}.$$

Levels of randomness

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The annealed setting:

Limit theorems for the ensemble-averaged spectral measure $\mathbb{E}\mu_n$:

$$\int f d(\mathbb{E}\mu_n) := \mathbb{E} \int f d\mu_n.$$

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The quenched setting:
Almost sure (or a.a.s.) bounds on the random variable $d(\mu_n, \nu)$

Distance to the ensemble average

Theorem (M.–Melcher)

Let $\{U_t^N\}_{N\in\mathbb{N},t\geq 0}$ *be such that for each N,* U_t^N *is a Brownian motion on* $\mathbb{U}(N)$*, with spectral measure* μ_t^N*.*

There is a constant $C > 0$ *such that with probability one, for all* $N \in \mathbb{N}$ *sufficiently large and for all* $t > 0$*,*

$$W_1(\mu_t^N, \mathbb{E}\mu_t^N) \leq C\left(\frac{t}{N^2}\right)^{1/3}.$$

Moreover, for all $N \in \mathbb{N}$ *sufficiently large and all* $t \geq 8(\log N)^2$*,*

$$W_1(\mu_t^N, \mathbb{E}\mu_t^N) \leq \frac{C}{N^{2/3}}.$$

Paths of measures

Theorem (M.–Melcher)

There are constants c, C such that for any $T \geq 0$ and for all $x \geq c\frac{T^{2/5}\log(N)}{N^{2/5}}$,

$$\mathbb{P}\left(\sup_{0\leq t\leq T} W_1(\mu_t^N, \nu_t) > x\right) \leq C\left(\frac{T}{x^2}+1\right)e^{-\frac{N^2x^2}{T}}.$$

In particular, with probability one for N sufficiently large

$$\sup_{0\leq t\leq T} W_1(\mu_t^N, \nu_t) \leq c\frac{T^{2/5}\log(N)}{N^{2/5}}.$$

Concentration of measure

A standard argument using the fact that $\mathbb{U}(N)$ has nonnegative Ricci curvature implies that for $F : \mathbb{U}(N) \subseteq \mathbb{M}_N \to \mathbb{R}$ a 1-Lipschitz function with respect to $\sqrt{N}\|\cdot\|_{H.S.}$,

$$\mathbb{P}\left[\left|F(U_t^N) - \mathbb{E}F(U_t^N)\right| > r\right] \leq 2e^{-\frac{r^2}{t}}.$$

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On the other hand, at stationarity (i.e., if U is distributed according to Haar measure on $\mathbb{U}(N)$), a clever coupling argument shows that

$$\mathbb{P}\left[\left|F(U) - \mathbb{E}F(U)\right| > r\right] \leq 2e^{-cr^2}.$$

Concentration of measure

Proposition (M.–Melcher)

There is a constant $C > 0$ *such that for all* $N \in \mathbb{N}$, $t \geq 8(\log N)^2$ *and* $r > 0$, *and all 1-Lipschitz functions* $F : \mathbb{U}(N) \to \mathbb{R}$,

$$\mathbb{P}\left(|F(U_t) - \mathbb{E}F(U_t)| > r\right) \leq Ce^{-\frac{r^2}{4}}.$$

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The proof takes advantage of

$z_t = e^{\frac{ib_t}{N}}$
(b_t a standard BM on $\mathbb{R}$)

V_t a standard BM on $\mathbb{SU}(N)$
(independent of b_t)

$\Longrightarrow$ $z_t V_t$ is a standard BM on $\mathbb{U}(N)$.

Concentration of the empirical spectral measure

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$$\implies \quad \mathbb{P}[W_1(\mu_t^N, \nu_t) > \mathbb{E}W_1(\mu_t^N, \nu_t) + r] \leq Ce^{-\frac{cN^2 r^2}{(t)}}.$$

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$\rightsquigarrow$ To show $W_1(\mu_t^N, \nu_t)$ is typically small, it's enough to show that $\mathbb{E}W_1(\mu_t^N, \nu_t)$ is small.

Average distance to average

One approach: consider the stochastic process

$$X_f := \int f d\mu_t^N - \mathbb{E} \int f d\mu_t^N.$$

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The concentration inequality for Lipschitz functions of U_t^N implies that $\{X_f\}_f$ satisfies a sub-Gaussian increment condition:

$$\mathbb{P}\left[|X_f - X_g| > r\right] \leq 2e^{-\frac{cN^2 r^2}{|f-g|_{L}^2(t)}}.$$

Dudley's entropy bound together with approximation theory, truncation arguments, etc., leads to the bound

$$\mathbb{E} W_1(\mu_t^N, \mathbb{E}\mu_t^N)$$
$$= \mathbb{E}\left(\sup_{|f|_L \leq 1} X_f\right) \leq C \begin{cases} \left(\frac{t}{N^2}\right)^{1/3}, & \text{all } t > 0; \\ \left(\frac{1}{N^2}\right)^{1/3}, & t \geq 8(\log(N))^2 \end{cases}.$$

Theorem (M.–Melcher)

Let μ_t^N be the spectral measure of U_t, where $\{U_t\}_{t \geq 0}$ is a Brownian motion on $\mathbb{U}(n)$ with $U_0 = I$.
For any $t, x > 0$,

$$\mathbb{P}\left(W_1(\mu_t^N, \mathbb{E}\mu_t^N) > c\left(\frac{t}{N^2}\right)^{1/3} + x \right) \leq 2e^{-\frac{N^2 x^2}{t}}.$$

For $x > 0$ and $t \geq 8(\log(N))^2$,

$$\mathbb{P}\left(W_1(\mu_t^N, \mathbb{E}\mu_t^N) > c\left(\frac{1}{N^2}\right)^{1/3} + x \right) \leq 2e^{-cN^2 x^2}.$$

Almost sure bounds on $W_1(\mu_t^N, \mathbb{E}\mu_t^N)$ are immediate from the Borel–Cantelli lemma.

Convergence of $\mathbb{E}\mu_t^N$ to ν_t

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Given $f : \mathbb{S}^1 \to \mathbb{R}$ a 1-Lipschitz function, let

$$S_m(z) := \sum_{|k|<m} \hat{f}(k) z^k.$$

Convergence of $\mathbb{E}\mu_t^N$ to ν_t

Given $f : \mathbb{S}^1 \to \mathbb{R}$ a 1-Lipschitz function, let

$$S_m(z) := \sum_{|k|<m} \hat{f}(k) z^k.$$

Observe that

$$\int z^k d\mu_t^N(z) = \frac{1}{N}\left[\mathrm{Tr}((U_t)^k)\right],$$

so that

$$\left|\int S_m \, d\mathbb{E}\mu_t^N - \int S_m \, d\nu_t\right| = \left|\sum_{1\le|k|<m} \hat{f}(k)\left(\frac{1}{N}\mathbb{E}[\mathrm{Tr}(U_t^k)] - \int z^k \, d\nu_t\right)\right|$$

$$\le \sum_{1\le|k|<m} \frac{\pi}{2k}\left|\frac{1}{N}\mathbb{E}[\mathrm{Tr}(U_t^k)] - \int z^k \, d\nu_t\right|.$$

Convergence of $\mathbb{E}\mu_t^N$ to ν_t

Collins–Dahlqvist–Kemp '18:

$$\left|\frac{1}{N}\mathbb{E}[\mathrm{Tr}(U_t^k)] - \int z^k \, d\nu_t\right| \le \frac{t^2 k^4}{N^2}.$$

Convergence of $\mathbb{E}\mu_t^N$ to ν_t

Collins–Dahlqvist–Kemp '18:

$$\left|\frac{1}{N}\mathbb{E}[\mathrm{Tr}(U_t^k)] - \int z^k d\nu_t\right| \leq \frac{t^2 k^4}{N^2}.$$

Using this estimate together with the classical fact that

$$\|f - S_m\|_\infty \leq C\left(\frac{\log(m)}{m}\right)$$

and optimizing over m leads to

$$W_1(\mathbb{E}\mu_t^N, \nu_t) \leq C\frac{t^{2/5}\log N}{N^{2/5}}.$$

Convergence of paths: Continuity of $\{\mathbb{E}\mu_t^N\}_{t\geq 0}$:

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$$\begin{aligned} W_1(\mathbb{E}\mu_t^N, \mathbb{E}\mu_s^N) &= \sup_{|f|_L \leq 1} \mathbb{E}\left[\int f d\mu_t^N - \int f d\mu_s^N\right] \\ &\leq \frac{\mathbb{E}\|U_t - U_s\|_N}{N} \\ &= \frac{\mathbb{E}\|I_N - U_{t-s}\|_N}{N}. \end{aligned}$$

Convergence of paths: Continuity of $\{\mathbb{E}\mu_t^N\}_{t\geq 0}$:

$$W_1(\mathbb{E}\mu_t^N, \mathbb{E}\mu_s^N) = \sup_{|f|_L \leq 1} \mathbb{E}\left[\int f d\mu_t^N - \int f d\mu_s^N\right]$$
$$\leq \frac{\mathbb{E}\|U_t - U_s\|_N}{N}$$
$$= \frac{\mathbb{E}\|I_N - U_{t-s}\|_N}{N}.$$

General properties of Brownian motion on manifolds together with estimates on volume ratios of balls in $\mathbb{U}(N)$ yield a concentration inequality for $\|I_N - U_{t-s}\|_N$ and ultimately,

$$W_1(\mathbb{E}\mu_t^N, \mathbb{E}\mu_s^N) \leq 3\sqrt{t-s} + \frac{1}{N}.$$

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Using

- the established convergence of $\mathbb{E}\mu_t^N$ to ν_t
- the continuity of $\{\mathbb{E}\mu_t^N\}_{t\geq 0}$

if $0 < s < t$,

$$W_1(\nu_t, \nu_s) \leq W_1(\nu_t, \mathbb{E}\mu_t^N) + W_1(\nu_s, \mathbb{E}\mu_s^N) + W_1(\mathbb{E}\mu_t^N, \mathbb{E}\mu_s^N)$$
$$\leq C\frac{(t^{2/5} + s^{2/5})\log N}{N^{2/5}} + 3\sqrt{t-s} + \frac{1}{N}.$$

Letting $N \to \infty$ yields

$$W_1(\nu_t, \nu_s) \leq 3\sqrt{t-s}.$$

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Fix $T > 0$, let $m \in \mathbb{N}$, and for $j = 1, \ldots, m$, let $t_j := \frac{jT}{m}$.

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Fix $T > 0$, let $m \in \mathbb{N}$, and for $j = 1, \ldots, m$, let $t_j := \frac{jT}{m}$. If $x > 9\sqrt{\frac{T}{m}}$

$$\mathbb{P}\left(\sup_{0\leq t\leq T} W_1(\mu_t^N, \nu_t) > x\right)$$

$$\leq \mathbb{P}\left(\max_{1\leq j\leq m} \sup_{|t-t_j|<\frac{T}{m}} W_1(\mu_t^N, \mu_{t_j}^N) > \frac{x}{3}\right)$$

$$+ \mathbb{P}\left(\max_{1\leq j\leq m} W_1(\mu_{t_j}^N, \nu_{t_j}) > \frac{x}{3}\right)$$

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$$
\begin{aligned}
&\mathbb{P}\left(\sup_{0\le t\le T} W_1(\mu_t^N, \nu_t) > x\right) \\
&\quad \le \mathbb{P}\left(\max_{1\le j\le m} \sup_{|t-t_j|<\frac{T}{m}} W_1(\mu_t^N, \mu_{t_j}^N) > \frac{x}{3}\right) \\
&\qquad\qquad + \mathbb{P}\left(\max_{1\le j\le m} W_1(\mu_{t_j}^N, \nu_{t_j}) > \frac{x}{3}\right) \\
&\quad \le m\mathbb{P}\left(\sup_{|t|<\frac{T}{m}} \|I_N - U_t\| > \frac{Nx}{3}\right) + m \max_{1\le j\le m} \mathbb{P}\left(W_1(\mu_{t_j}^N, \nu_{t_j}) > \frac{x}{3}\right).
\end{aligned}
$$

Convergence of paths

Theorem (M.–Melcher)

Let $T \geq 0$. There are constants c, C such that for all $x \geq c\frac{T^{2/5}\log(N)}{N^{2/5}}$,

$$\mathbb{P}\left(\sup_{0\leq t\leq T} W_1(\mu_t^N, \nu_t) > x\right) \leq C\left(\frac{T}{x^2}+1\right)e^{-\frac{N^2x^2}{T}}.$$

In particular, with probability one for N sufficiently large

$$\sup_{0\leq t\leq T} W_1(\mu_t^N, \nu_t) \leq c\frac{T^{2/5}\log(N)}{N^{2/5}}.$$

Thank you.