Semialgebraicity and constructions with convex bodies

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Online Asymptotic Geometric Analysis Seminar
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MAX-PLANCK-GESELLSCHAFT

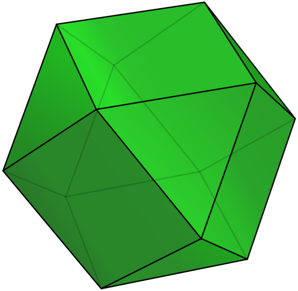
1. (Semi)algebraic geometry and convexity

2. Fiber convex bodies

3. Intersection bodies

(Semi)algebraic geometry and convexity

How do we generalize polytopes?



finite representation

f-vector

linear algebra

⇝ **semialgebraic convex bodies**

Semialgebraic convex bodies

A subset $K \subset \mathbb{R}^d$ is a convex body if it is convex, compact and non-empty (today we will also assume full dimensional).

Definition

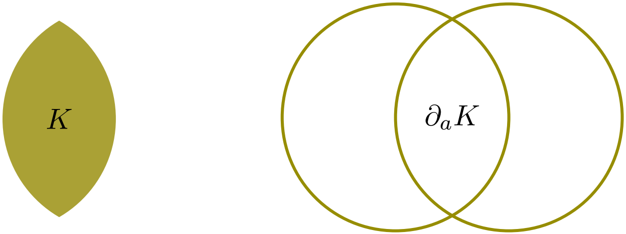
A convex body $K \subset \mathbb{R}^d$ is semialgebraic if it is a semialgebraic set: a finite Boolean combination of polynomial inequalities.

The topological boundary ∂K of K is also a semialgebraic set. Also intersections, projections, the dual/polar body, the Minkowski sum,...

Definition

The algebraic boundary of $K \subset \mathbb{R}^d$, denoted by $\partial_a K \subset \mathbb{C}^d$, is the closure of ∂K with respect to the Zariski topology. In other words, it is the smallest complex algebraic variety that contains ∂K.

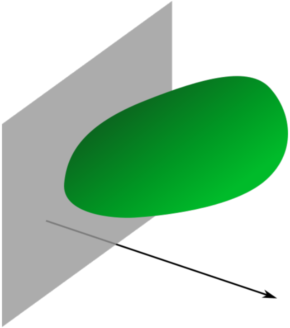
Example



Fiber convex bodies

Setting

We work in $\mathbb{R}^{n+m}$. Consider $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ the projection onto the first n coordinates, and let $K \subset \mathbb{R}^{n+m}$ be a convex body.



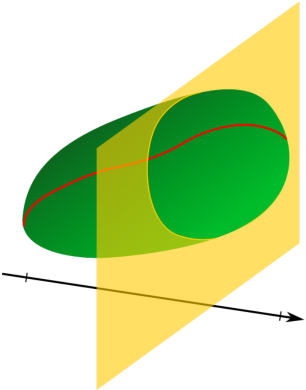
Goal: construct a convex body of $\mathbb{R}^m$.

Definition

The fiber body of K, with respect to π, is given by

$$\Sigma_\pi K = \left\{ y = \int_{\pi(K)} \gamma(x)\, \mathrm{d}x \mid \gamma \text{ measurable section} \right\}$$

with $\gamma : \pi(K) \to K$ such that $\pi \circ \gamma(x) = x$.



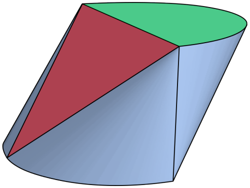
K_x is the fiber of K over x.

Some remarks

- **Continuity is not enough:**

$$\left\{ y = \int_{\pi(K)} \gamma(x)\,\mathrm{d}x \mid \gamma \text{ continuous section} \right\} \subset \Sigma_\pi K$$

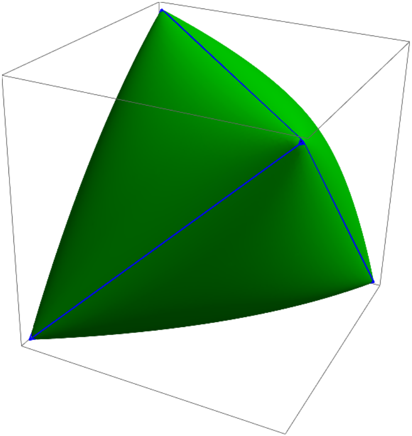
If K is a polytope, the inclusion is an equality!



In general? It is a strict inclusion.

- **Support function:**

$$h_{\Sigma_\pi K}(u) = \int_{\pi(K)} h_{K_x}(u)\,\mathrm{d}x.$$

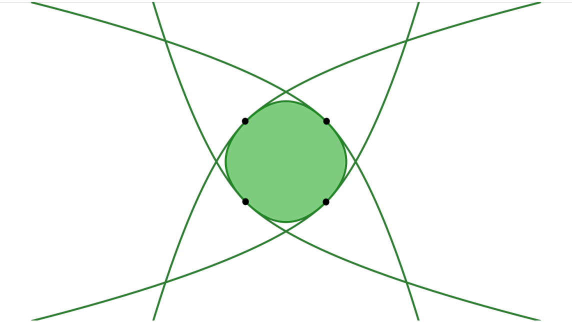


$$K = \left\{(x,y,z) \in [-1,1]^3 \mid x^2 + y^2 + z^2 - 2xyz \leq 1\right\}$$

Let's integrate its support function:

$$
\begin{aligned}
h_{\Sigma_\pi K}(y,z) &= \int_{-1}^{1} h_{K_x}(y,z)\,\mathrm{d}x \\
&= \int_{-1}^{1} \sqrt{y^2+z^2+2xyz}\,\mathrm{d}x \\
&= \frac{1}{3yz}\left(|y+z|^3 - |y-z|^3\right).
\end{aligned}
$$

$$\Sigma_\pi K = \Big\{ (y,z) \in \mathbb{R}^2 \,|\, 3y^2 + 8z - 16 \leq 0, 3y^2 - 8z - 16 \leq 0, \\ 3z^2 + 8y - 16 \leq 0, 3z^2 - 8y - 16 \leq 0 \Big\}$$



Zoonoids

Zonoid (centered at the origin): limit of zonotopes $\sum_{i=1}^{\infty}[-z_i, z_i]$.

Theorem (Vitale - 1991)

A convex body $K \subset \mathbb{R}^d$ is a zonoid if and only if there is a random vector $X \in \mathbb{R}^d$ with $\mathbb{E}\|X\| < \infty$ such that for all $u \in \mathbb{R}^d$

$$h_K(u) = \frac{1}{2}\mathbb{E}\,|\langle u, X\rangle|\,.$$

Procedure: $K \subset \mathbb{R}^{n+m}$ zonoid

$$\downarrow$$

$X \in \mathbb{R}^{n+m}$ random vector associated to K

$$\downarrow$$

Consider the function $F_\pi : (\mathbb{R}^{n+m})^{n+1} \to \mathbb{R}^m$ that maps the point $(x_1 + y_1, \ldots, x_{n+1} + y_{n+1})$ to

$$\frac{1}{(n+1)!} \sum_{i=1}^{n+1} (-1)^{n+1-i} (x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_{n+1}) y_i$$

and let $Y := F_\pi \left(X^{(1)}, \ldots, X^{(n+1)} \right) \in \mathbb{R}^m$

$$\downarrow$$

$Z(Y)$, the zonoid of $\mathbb{R}^m$ associated to Y

Theorem (Mathis, M. - 2021)

Let K be the zonoid associated to the random vector X, then

$$h_{\Sigma_\pi K}(u) = \frac{1}{2}\mathbb{E}|\langle u, Y\rangle|$$

where $Y := F_\pi\left(X^{(1)}, \dots, X^{(n+1)}\right)$ and $X^{(1)}, \dots, X^{(n+1)}$ are i.i.d. copies of X.

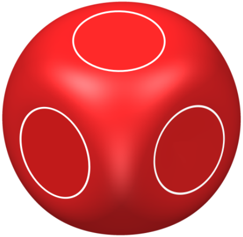
In particular, if we consider the zonotope

$$K = \sum_{i=1}^{s}[-z_i, z_i] \subset \mathbb{R}^{n+m}$$

then its fiber body is again a zonotope, given by

$$(n+1)! \sum_{1\leq i_1<\dots<i_{n+1}\leq s} \Big[-F_\pi\left(z_{i_1}, \dots, z_{i_{n+1}}\right), F_\pi(z_{i_1}, \dots, z_{i_{n+1}})\Big].$$

$$K = \{x = 0, y^2 + z^2 \leq 1\} + \{y = 0, x^2 + z^2 \leq 1\} + \{z = 0, x^2 + y^2 \leq 1\}$$

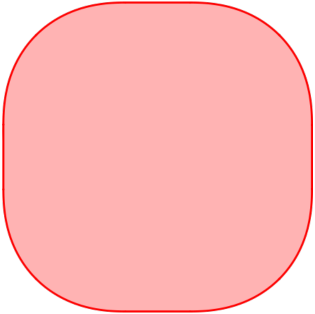


With respect to the projection $\pi(x, y, z) = x$, the fiber body of the dice is

$$\Sigma_\pi K = D_3 + \frac{\pi}{4}(S_2 + S_3) + \frac{1}{2}\Lambda$$

where Λ is the convex body whose support function is given by

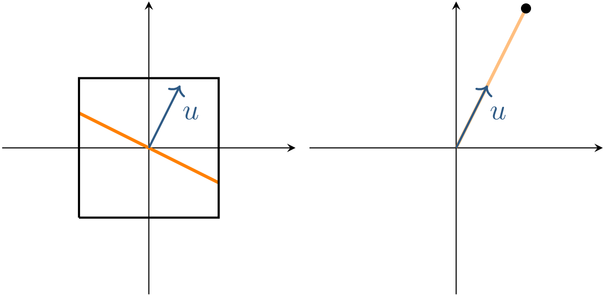
$$h_\Lambda(u, v) = \frac{1}{2}\int_0^\pi \sqrt{\cos(\theta)^2 (u)^2 + \sin(\theta)^2 (v)^2}\, \mathrm{d}\theta.$$



Intersection bodies

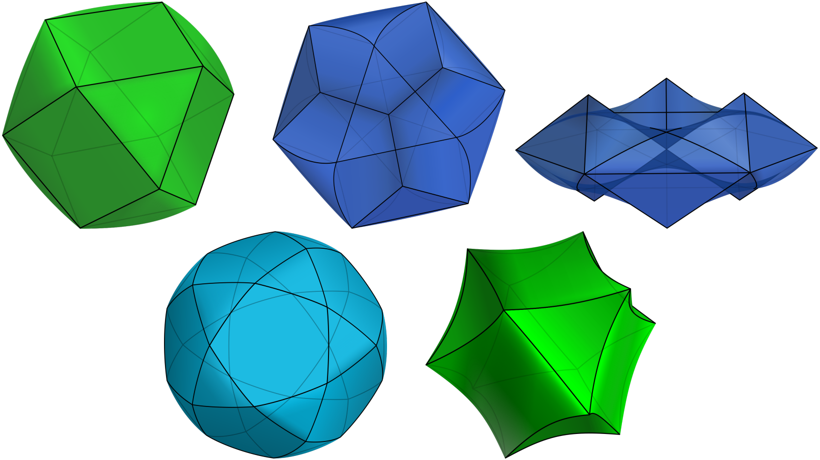
Let K be a convex body in $\mathbb{R}^d$. Its intersection body is defined to be the set $IK = \{x \in \mathbb{R}^d \mid \rho_{IK}(x) \geq 1\}$, where the radial function (restricted to the sphere) is

$$\rho_{IK}(u) = \mathrm{vol}_{d-1}(K \cap u^{\perp}).$$

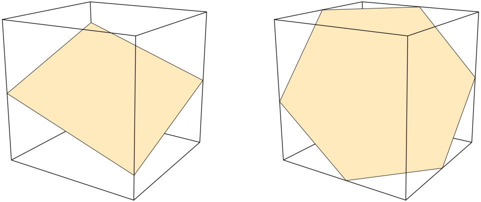


We will focus here on the intersection body of a polytope P.

Intersection bodies ⟩ gallery



Let P be the three dimensional cube centered at the origin.



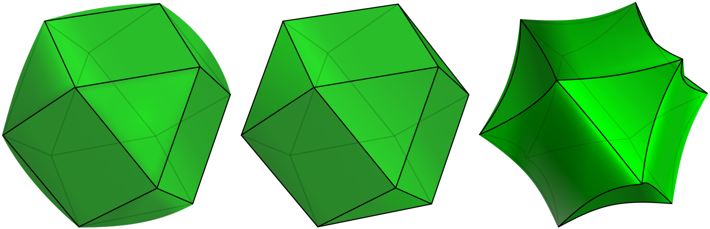
Define the zonotope associated to P as

$$Z(P) = \sum_{v \text{ is a vertex of } P} [-v, v].$$

Lemma

The maximal cones of its normal fans determine the regions of IP. Equivalently, the facets of $Z(P)^\circ$ determine the regions.

Intersection bodies 〉 the associated zonotope



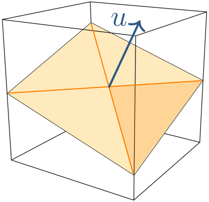
Theorem (Berlow, Brandenburg, M., Shankar - 2022)

Let $P \subseteq \mathbb{R}^d$ be a full-dimensional polytope. Then IP, the intersection body of P, is a semialgebraic starshaped set.

In particular, the radial function of IP is piecewise rational. The pieces are exactly the maximal cones of the normal fan of $Z(P)$.

Why?

Sketch of the proof:



$$\rho_{IP}(u) = \sum_{j=1}^{4} \mathrm{vol}_{d-1}(\Delta_j) = \frac{1}{(d-1)!} \sum_{j=1}^{4} |\det(M_j(u))|$$

$$= \frac{p(u)}{q(u)}$$

it is semialgebraic!

Degree bound

Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope with $f_1(P)$ edges. Then the degrees of the irreducible components of the algebraic boundary of IP are bounded from above by

$$f_1(P) - (d-1).$$

`https://mathrepo.mis.mpg.de/intersection-bodies`

Thank you!

- Fiber Convex Bodies, with L. Mathis, arXiv:2105.12406
- Intersection Bodies of Polytopes, with K. Berlow, M. C. Brandenburg, I. Shankar, Beitr Algebra Geom (2022)