Name: ______________________ GTID: ______________________

- Fill out your name and Georgia Tech ID number.
- This Midterm contains 8 pages. Please make sure no page is missing.
- The grading will be done on the scanned images of your test. Please write clearly and legibly.
- Answer the questions in the spaces provided. We will scan the front sides only by default. If you run out of room for an answer, continue on the back of the page and notify the proctor when handing in.
- Please write detailed solutions including all steps and computations.
- The duration of the Midterm is 75 minutes.

Good luck!

1. Determine whether each differential equation is linear or non-linear.

   (a) (2 points) $y'(t) = y^2 - y^3$
   ◯ linear $\surd$ **non-linear**

   (b) (2 points) $\dfrac{dy}{dt} = e^{y+t}$
   ◯ linear $\surd$ **non-linear**

   (c) (2 points) $\dfrac{dy}{dt} + \dfrac{1}{t}\, y = \ln t, t > 0$
   $\surd$ **linear** ◯ non-linear

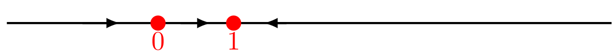
2. Consider the differential equation $y'(t) = y^2 - y^3$. Note that its critical points are $y = 0, 1$.

   (a) (5 points) Draw the phase line for the differential equation.

   (b) (6 points) Classify the critical points.

Solution:

(a) Note that $y^2 - y^3 = y^2(1 - y)$. In particular it is non-negative when $y \leq 1$ is less than 1 and non-positive when $y \geq 1$. Therefore y' is less than 0 when $y < 0$ and when $0 < y < 1$ and it is larger than 0 when $y > 1$. Since the critical points are 0 and 1 we get the following phase line.



(b) Since y' is positive on the intervals $(-\infty, 0)$ and $(0, 1)$ the point $y = 0$ is semi-stable. Since y' is positive on the interval $(0, 1)$ and negative on the interval $(1, \infty)$, the point $y = 1$ is asymptotically stable.

3. (12 points) Solve using the method of integrating factors: $\dfrac{dy}{dt} + \dfrac{1}{t}\, y = \ln t$, $t > 0$.

Solution: Integrating factor is $e^{\int \frac{1}{t}dt} = e^{\ln t} = t.$ $\longrightarrow$ 3 points

Multiplying both sides of the DE by the integrating factor and rearranging the terms, we get

$\dfrac{d}{dt}(yt) = t \ln t \longrightarrow$ 3 points

Integrating both sides we get

$yt = \dfrac{t^2}{2}\ln t - \dfrac{t^2}{4} + c. \longrightarrow$ 3 points

Rearranging the terms, we get

$y = \dfrac{t}{2}\ln t - \dfrac{t}{4} + \dfrac{c}{t}. \longrightarrow$ 3 points

4. (11 points) Solve the following exact equation (recall that the equation is called exact if for some continuously differentiable function ϕ of two variables, the equation is of the form $\frac{\partial}{\partial x}\phi(x,y)+\frac{\partial}{\partial y}\phi(x,y)y'=0$).

$$(2xy^2+2y)+(2x^2y+2x)y'=0,\ t\in\mathbb{R}.$$

Solution: Note that letting $\phi(x,y) = x^2y^2 + 2xy$, we get that the equation is equivalent to $\frac{\partial}{\partial x}\phi(x,y)+\frac{\partial}{\partial y}\phi(x,y)y'=0$, and therefore, $\phi(x,y(x)) = c_1$ where c_1 is a constant. In other words,

$$x^2y(x)^2+2xy(x)=c_1,$$

or equivalently

$$(xy(x)+1)^2=C,$$

where we denote $C = c_1 + 1$. This leads to the answer

$$y(x)=\frac{c}{x},$$

where c is some constant.

5. (20 points) Consider the following system of equations:

$$\begin{cases} x_1' = x_1 + x_2, \\ x_2' = -x_1 + x_2. \end{cases}$$

Find the general solution to this system and express your answer only using real-valued functions.

Solution: Eigenvalues: $|\ \mathbf{A} - \lambda \mathbf{I}\ | = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = \mathbf{0}$

$\iff \lambda_1 = 1+i, \lambda_2 = 1-i \longrightarrow$ 8 points

Eigenvectors for $\lambda_1 = 1+i : (\mathbf{A} - (1+i)\mathbf{I})\mathbf{X} = \mathbf{0} \iff$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff v_2 = iv_1$$

$$\text{When } v_1 = 1, \text{ the eigenvector is } \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} \longrightarrow \text{4 points}$$

Note that $\mathrm{Re}(\mathbf{v}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathrm{Im}(\mathbf{v}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So, the general solution is :

$$\mathbf{X} = e^t \left(c_1 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t \right) + c_2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t \right) \right) \text{ or}$$

$$\iff \mathbf{X} = c_1 e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \longrightarrow \text{8 points}$$

6. (a) (12 points) Find the general solution of $\mathbf{X}' = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \mathbf{X}$.

   (b) (4 points) Draw the phase portrait and characterize the origin.

   (c) (4 points) Solve the initial value problem: $\mathbf{X}' = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \mathbf{X}$, with the initial condition $\mathbf{X}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Solution:

(a) First, we find the eigenvalues of $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ as the solutions of the charateristic equation

$$(2-\lambda)^2 - 9 = 0,$$

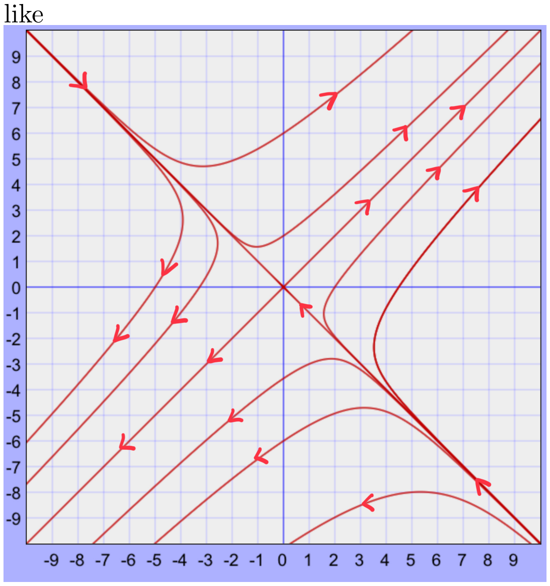
so $\lambda_1 = -1$ and $\lambda_2 = 5$.

The first eigenvector $v_1 = \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$ satisfies $v_{12} = -v_{11}$, so $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The second eigenvector $v_2 = \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$ satisfies $v_{12} = v_{11}$, thus $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Thus the general solution is $\mathbf{X}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, for some constants $c_1, c_1 \in \mathbb{R}$.

(b) Since $\lambda_1 < 0 < \lambda_2$, the origin is a saddle point and therefore unstable. The phase portrait looks like



(c) Using the general solution, we need to solve the linear system

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \mathbf{X}(0) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Using our favorite method to solve linear systems, we obtain $c_1 = c_2 = 1$. Therefore, the solution to the initial value problem is

$$\mathbf{X}(t) = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

7. (20 points) Solve $\mathbf{X}' = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, assuming that $X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Solution: We first find an equilibrium point for the system of differential equations. Note that

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_1 + x_2 + 1 \end{pmatrix}.$$

Setting this to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ we see that $x_1 = 0$ and $x_2 = -1$. Therefore $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ is an equilibrium point. Next we solve the corresponding homogenous system $\mathbf{X}' = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \mathbf{X}$. Note that this is equivalent to

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_1 + x_2 \end{pmatrix}.$$

Now $x_1' = x_1 \implies x_1 = C_1 e^t$. Plugging the value of x_1 into the second equation gives $x_2' = 2C_1 e^t + x_2 \implies x_2' - x_2 = 2C_1 e^t$. This is a first-order linear ODE which can be solve using integrating factors. In this case $I = e^{\int -1 \, dt} = e^{-t}$ and so

$$x_2' - x_2 = 2C_1 e^t \iff (e^{-t} x_2)' = 2C_1 \iff x_2 = e^t \left(2C_1 t + C_2\right)$$

Therefore the solution to the homogenous system is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} C_1 e^t \\ 2C_1 t e^t + C_2 e^t \end{pmatrix} = C_2 e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_1 e^t \begin{pmatrix} 1 \\ 2t \end{pmatrix}$$

Therefore the general solution to the original system (without the initial condition) is

$$\mathbf{X}(t) = C_2 e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_1 e^t \begin{pmatrix} 1 \\ 2t \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Now, we look at the IVP. We solve for C_1, C_2. We have

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{X}(0) = \begin{pmatrix} 0 \\ C_2 \end{pmatrix} + \begin{pmatrix} C_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 - 1 \end{pmatrix}.$$

Therefore $C_1 = 0$ and $C_2 = 1$ and so the solution to the IVP is

$$\mathbf{X}(t) = e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Alternative solution: We can also use generalized eigenvectors directly to solve the homogeneous system. Notice that the eigenvector can be chosen to be $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The system

$$\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

yields the generalized eigenvector $\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$.

8. *** (Bonus: 10 points) solve the following Bernoulli equation:

$$y' = y + \sqrt{y},\ t \in \mathbb{R}.$$

Solution: We have that

$$y' = y + y^{1/2} \iff y^{-1/2}y' - y^{1/2} - 1 = 0.$$

Substituting $u = y^{1/2}$ we have that $u' = y^{-1/2}(1/2)y' \implies y' = 2y^{1/2}u'$. Substituting u for y in the differential equation gives

$$y^{-1/2}(2y^{1/2}u') - u - 1 = 0 \iff 2u' - u - 1 = 0 \iff u' = \frac{u+1}{2}.$$

Substituting $v = (u+1)/2$ gives $v' = u'/2 \implies u' = 2v'$. Substituting v for u then gives

$$2v' = v \iff v' = v/2.$$

Solving for v we have that

$$v' = v/2 \implies \frac{v'}{v} = \frac{1}{2} \implies \log|v| = \frac{t}{2} + C_1 \implies v = C_2e^{t/2}$$

Since $u = 2v - 1$ we have $u = C_3e^{t/2} - 1$. Finally, since $y = u^2$, we have that

$$y = (C_3e^{t/2} - 1)^2.$$