Name: ______________________ GTID: ______________________

- Fill out your name and Georgia Tech ID number.
- This exam contains 8 pages. Please make sure no page is missing.
- The grading will be done on the scanned images of your test. Please write clearly and legibly.
- Answer the questions in the spaces provided. We will scan the front sides only by default. If you run out of room for an answer, continue on the back of the page and notify the TA when handing in.
- Please write detailed solutions including all steps and computations. Include worded explanations when necessary.
- The duration of the exam is 75 minutes.

Good luck!

1. (16 points) Verify that $y_1(x) = x$ and $y_2(x) = x\ln x$ form a fundamental set of solutions of

$$x^2y'' - xy' + y = 0$$

when $x > 0$.

Solution:

For $y_1 = x$, $y_1' = 1$ and $y_1'' = 0$. So,

$$x^2y_1'' - xy_1' + y_1 = 0. \longrightarrow \text{5 points}$$

For $y_2 = x\ln x$, $y_2' = \ln x + 1$ and $y_2'' = \dfrac{1}{x}$. So,

$$x^2y_2'' - xy_2' + y_2 = 0. \longrightarrow \text{5 points}$$

$W(y_1, y_2) = \begin{vmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{vmatrix} = x \neq 0. \longrightarrow$ 5 points

Hence, $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of $x^2y'' - xy' + y = 0$ when $x > 0$. $\longrightarrow$ 1 point

2. (17 points) Find a second order differential equation whose general solution is

$$y = c_1 e^{2t}\cos(2t) + c_2 e^{2t}\sin(2t)$$

Solution:

Recall that the characteristic equation of the second order differential equation

$$ay'' + by' + cy = 0$$

is

$$a\lambda^2 + b\lambda + c = 0$$

and whose roots are

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

To find a differential equation whose solution is y, we pick a, b, c so that the roots of the chracteristic equation are complex conjugates with real part equal to 2 and imaginary part equal to 2. In particular we want

$$-\frac{b}{2a} = 2, \frac{\sqrt{b^2 - 4ac}}{2a} = 2i. \longrightarrow \text{6 point}$$

For simplicity we take $a = 1 \longrightarrow$ 2 point.

The first equation is then equivalent to $b = -4 \longrightarrow$ 3 point.

Squaring the second equation and solving for c we get $c = (b^2 + 16)/4 = 32/4 = 8 \longrightarrow$ 3 point. Therefore the differential equation

$$y'' - 4y' + 8y = 0 \longrightarrow \text{3 point}$$

has $y = c_1 e^{2t}\cos(2t) + c_2 e^{2t}\sin(2t)$ as its general solution.

3. (16 points) Find the Laplace transform, $Y(s) = \mathcal{L}\{y(t)\}$, of the function

$$y(t) = 4 + \sin(8t) - 2e^t \cos t$$

defined on the interval $t \geq 0$

Solution:

$$\begin{aligned}\mathcal{L}(4 + \sin 8t - 2e^t \cos t) &= \mathcal{L}(4) + \mathcal{L}(\sin 8t) - 2\mathcal{L}(e^t \cos t) \longrightarrow \text{6 point} \\ &= \frac{4}{s} + \frac{8}{s^2 + 64} - \frac{2(s-1)}{(s-1)^2 + 1} \longrightarrow \text{10 points}\end{aligned}$$

4. (17 Points) Find the inverse Laplace transform of $\dfrac{s}{s^2 - 2s + 2}$

Solution:

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{s^2-2s+2}\right\} &= \mathcal{L}^{-1}\left\{\frac{s}{(s-1)^2+1}\right\} \to \text{3 point} \\
&= \mathcal{L}^{-1}\left\{\frac{s-1+1}{(s-1)^2+1}\right\} \to \text{3 point} \\
&= \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} \to \text{3 point} \\
&= e^t \cos t + e^t \sin t \to \text{8 points}
\end{aligned}$$

5. (17 points) Find the Laplace transform of the solution of

$$y'' + 8y' = 0, y(0) = 4, y'(0) = -2.$$

Solution:

Applying the Laplace transform yields

$$\mathcal{L}\Big\{y''\Big\} + 8\mathcal{L}\Big\{y'\Big\} = \mathcal{L}\Big\{0\Big\} \to \text{3 point}$$

$\implies s^2Y(s) - sy(0) - y'(0) + 8sY(s) - 8y(0) = 0 \to$ 10 points

$\implies (s^2 + 8s)Y(s) = 4s + 30 \to$ 2 points

$\implies Y(s) = \dfrac{4s + 30}{s^2 + 8s} \to$ 2 point

6. (17 points) Using the Laplace transform, find the solution to

$$y^{(4)} - 9y = 0,$$

$$y(0) = 1,\ y'(0) = 0,\ y''(0) = -3,\ y'''(0) = 0.$$

Solution:

Note that letting $\mathcal{L}y = Y$ we get, from the properties of Laplace transform:

$$\mathcal{L}(y^{(4)} - 9y)|_s = s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - 9Y(s). \longrightarrow$$ 8 point

Using our IVP, together with the fact that our ODE implies that $\mathcal{L}(y^{(4)} - 9y)|_s = 0$, we get

$$(s^4 - 9)Y(s) - s^3 + 3s = 0. \longrightarrow$$ 3 point

We get that either $s^2 = 3$ or we cancel out $s^2 - 3$ to obtain

$$\mathcal{L}y(s) = Y(s) = \frac{s}{s^2+3}. \longrightarrow$$ 3 point

Therefore,

$$y(t) = \mathcal{L}^{-1}\left(\frac{s}{s^2+3}\right) = \cos(\sqrt{3}t). \longrightarrow$$ 3 point

We observe aposteriori that this function is continuous, bounded and does satisfy our IVP.

Answer: $y(s) = \cos(\sqrt{3}t)$.

7. (optional bonus 10 points)

   1. Explain why the Laplace transform of a piecewise-continuous function of exponential order tends to zero at infinity.
   2. Give an example of a continuous function whose Laplace transform does not tend to zero at infinity.

Solution:

1. If f is a function of exponential order then there exists constants $M \geq 0, K > 0, a$ such that $|f(t)| < Ke^{at}$ for all $t \geq M$. Take $Q := \sup_{0 \leq t \leq M} |f(t)|$ and suppose $s > a$. Then

$$\begin{aligned} |\mathcal{L}\{f\}(s)| &= \left| \int_0^\infty e^{-st} f(t)\ dt \right| \\ &\leq \int_0^\infty e^{-st} \left| f(t) \right|\ dt \\ &\leq \int_0^M e^{-st} |f(t)|\ dt + \int_M^\infty e^{-st} |f(t)|\ dt \\ &\leq Q \int_0^M e^{-st}\ dt + \int_M^\infty K e^{t(a-s)}\ dt \\ &\leq \frac{Q}{s} + \frac{e^{-M(s-a)}}{s-a}. \end{aligned}$$

   Both terms in the last line go to 0 as s goes to infinity so we conclude that the laplace transform tends to 0 as s tends to infinity. $\rightarrow$ 5 point

2. Take $f(t) = e^{t^2}$. Then

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} e^{t^2}\ dt \geq \int_s^\infty e^{-st} e^{t^2}\ dt \geq \int_s^\infty 1\ dt = \infty.$$

   In particular the Laplace transform of f is equal to infinity for all s. $\rightarrow$ 5 point