

Novel view on classical convexity theory

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AGA Online Seminar, May 2020

Introduction

Start of the subject in a joint work with Emanuel Milman

Denote by $B(x, r)$ the Euclidean ball centered at x and has radius $r > 0$.

Let \mathcal{B}_0 be the family of all balls which contain 0. In other words $B(x, r) \in \mathcal{B}_0$ if and only if $|x| \leq r$, where $|x|$ is the Euclidean norm of x .

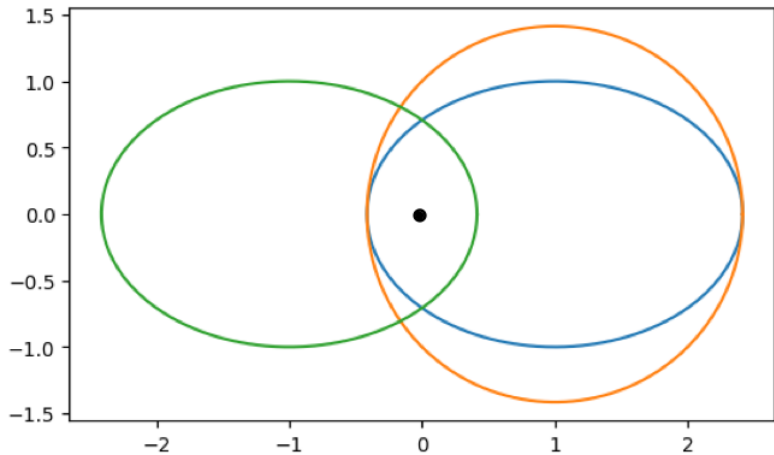
Write $B_x = B\left(\frac{x}{2}, \frac{|x|}{2}\right)$, i.e. the ball that has $[0, x]$ as its diameter. We call B_x a **petal**. Also write $B_2^n = B(0, 1)$ for the unit ball.

Definition

A **flower** is any set of the form $F = \bigcup_{\alpha} B_{\alpha}$ for a collection of balls $\{B_{\alpha}\}_{\alpha} \subseteq \mathcal{B}_0$. We denote the family of all flowers by \mathcal{F} .

Equivalent Definition

A flower is $F = \bigcup_{x \in A} B_x$ – a union of petals.



Every flower F uniquely represents a pair (K, K°) where $K \in \mathcal{K}_0$, i.e. a closed convex set containing 0, and K° is the canonical dual of K . More precisely, we call K the **core** of F if

$$K = \{x \in \mathbb{R}^n : B_x \subseteq F\}.$$

Let φ be **spherical inversion**, i.e. $\varphi(x) = \frac{x}{|x|^2}$ for $x \neq 0$. For any star body A (i.e. such that $\lambda A \subseteq A$ for all $0 \leq \lambda \leq 1$), define the co-image of φ by

$$\text{co}\varphi(A) = \overline{\{x : x \notin \varphi(A)\}}$$

(i.e. the closure of the complement $\varphi(A)^c$). Note that the closure is always radial. Then consider the set $T := \text{co}\varphi(F)$.

Fact

For any flower F , the bodies K and T from the above construction belong to \mathcal{K}_0 , and $T = K^\circ$.

Every $K \in \mathcal{K}_0$ is the core of a unique flower which we denote by $F = K^\clubsuit$. The map $\clubsuit : \mathcal{K}_0 \rightarrow \mathcal{F}$ is called the flower map, and we denote its inverse (the core operation) by $K = F^{-\clubsuit}$. We therefore have one to one and onto maps

$$\mathcal{K}_0 \xrightarrow{\clubsuit} \mathcal{F} \xrightarrow{\text{co}\varphi} \mathcal{K}_0,$$

and their composition is exactly the duality map: $\text{co}\varphi(K^\clubsuit) = K^\circ$. So, every flower F “sees” simultaneously a convex body $K = F^{-\clubsuit}$ and its dual $K^\circ = \text{co}\varphi(F)$. Since $\text{co}\varphi$ is an involution, we obtain an equivalent definition of the class of flowers \mathcal{F} :

Equivalent Definition

$\mathcal{F} = \text{co}\varphi(\mathcal{K}_0)$, i.e. **flowers are the complements of inversions of convex bodies.**

Theorem (V. M and L. Rotem, [Essentially Artstein–M. or Böröczky–Schneider])

1. Let $f : \mathcal{F} \rightarrow \mathcal{K}_0$ be a one to one and onto map such that f and f^{-1} preserve the order of inclusion (i.e. $F_1 \subseteq F_2$ if and only if $f(F_1) \subseteq f(F_2)$). Then there exists an invertible linear map $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $f(F) = u(F^{-\clubsuit})$.
2. Let $g : \mathcal{F} \rightarrow \mathcal{K}_0$ be a one to one and onto map such that g and g^{-1} reverse the order of inclusion (i.e. $F_1 \subseteq F_2$ if and only if $g(F_1) \supseteq g(F_2)$). Then there exists an invertible linear map $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $g(F) = u(\text{co}\varphi(F))$.

We also present a fourth equivalent definition:

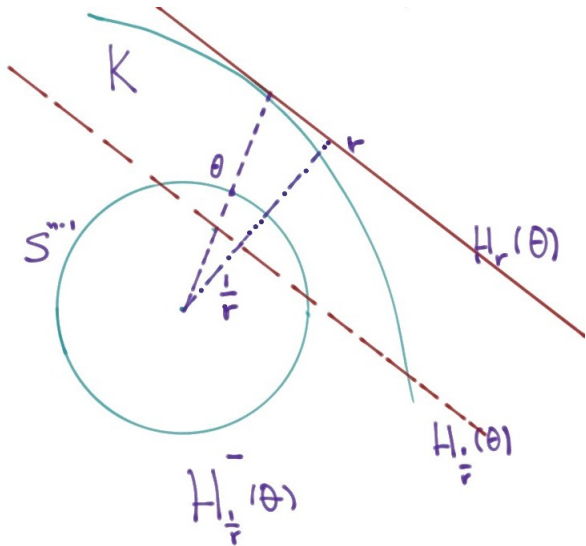
Let $h_K : S^{n-1} \rightarrow [0, \infty]$ be the supporting functional of a convex body $K \in \mathcal{K}_0$. (considered on $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, not as a 1-homogeneous function on \mathbb{R}^n).

Let F be the star body with radial function $r_F(\theta) = h_K(\theta)$ for all $\theta \in S^{n-1}$. Then F is a flower and $F = K^\clubsuit$. As the converse is also true:

Equivalent (4th) Definition

Flowers are exactly the star bodies whose radial function is convex (as a function on the sphere, meaning its 1-homogeneous extension is convex on \mathbb{R}^n).

This last description of the flower map \clubsuit is very useful in the different computations and constructions we will describe. It was actually our original definition.



Now we discuss flower mixed volumes. For any collection of flowers $\{F_i\}_{i=1}^m$ and positive numbers $\{\lambda_i\}_{i=1}^m$, construct a new flower by

$$\mathcal{G} = \mathcal{G}(\{F_i\}_i, \{\lambda_i\}_i) = \left(\sum_{i=1}^m \lambda_i F_i^{-\clubsuit} \right)^{\clubsuit}.$$

Then $|\mathcal{G}|$, the volume of \mathcal{G} , is a homogeneous polynomial of degree n :

$$|\mathcal{G}| = \sum_{1 \leq i_1, i_2, \dots, i_n \leq m} V(F_{i_1}, F_{i_2}, \dots, F_{i_n}) \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n},$$

where as usual we take the coefficients $V(F_{i_1}, F_{i_2}, \dots, F_{i_n})$ to be invariant with respect to permutations of their arguments.

We call these coefficients **flower mixed volumes**. For the cores $K_i = F_i^{-\clubsuit}$ we also set

$$V_{\clubsuit}(K_1, K_2, \dots, K_n) = V(F_1, F_2, \dots, F_n).$$

An explicit formula for these numbers is

$$V_{\clubsuit}(K_1, K_2, \dots, K_n) = |B_2^n| \cdot \int_{S^{n-1}} \prod_{i=1}^n h_{K_i}(\theta) d\sigma(\theta).$$

We have an elliptic type \clubsuit -Alexandrov Fenchel inequality

$$V_{\clubsuit}(K_1, K_2, \dots, K_n)^2 \leq V_{\clubsuit}(K_1, K_1, K_3, \dots, K_n) V_{\clubsuit}(K_2, K_2, K_3, \dots, K_n).$$

Let us set

$$W_{\clubsuit, i} = V_{\clubsuit}(\underbrace{K, K, \dots, K}_{n-i \text{ times}}, \underbrace{B_2^n, B_2^n, \dots, B_2^n}_i).$$

Then we have a Kubota type formula

$$W_{\clubsuit, n-i}(K) = c_{n,i} \cdot \int_{G_{n,i}} |(\text{Proj}_E K)^{\clubsuit}| d\mu(E),$$

and the inequalities (Alexandrov's inequalities and \clubsuit -Alexandrov inequalities):

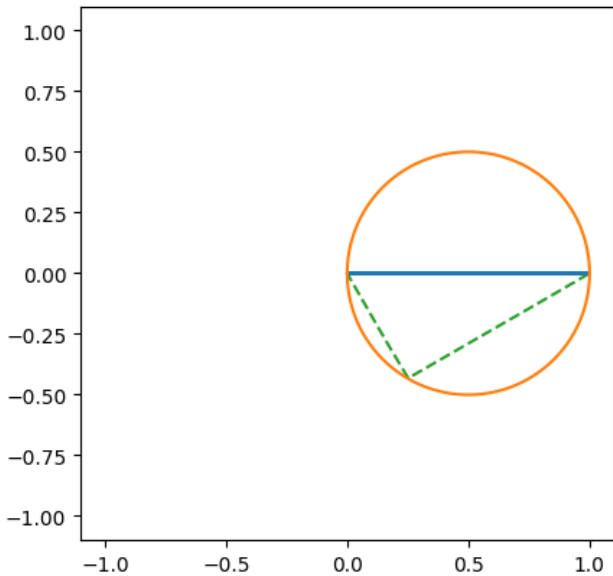
$$\begin{aligned} \left(\frac{|K|}{\omega_n}\right)^{\frac{1}{n}} &\leq \left(\frac{W_1(K)}{\omega_n}\right)^{\frac{1}{n-1}} \leq \dots \leq \frac{W_{n-1}(K)}{\omega_n} = \frac{W_{\clubsuit, n-1}(K)}{\omega_n} \\ &\leq \left(\frac{W_{\clubsuit, n-2}(K)}{\omega_n}\right)^{\frac{1}{2}} \leq \dots \leq \left(\frac{W_{\clubsuit, 1}(K)}{\omega_n}\right)^{\frac{1}{n-1}} \leq \left(\frac{|K^{\clubsuit}|}{\omega_n}\right)^{\frac{1}{n}}. \end{aligned}$$

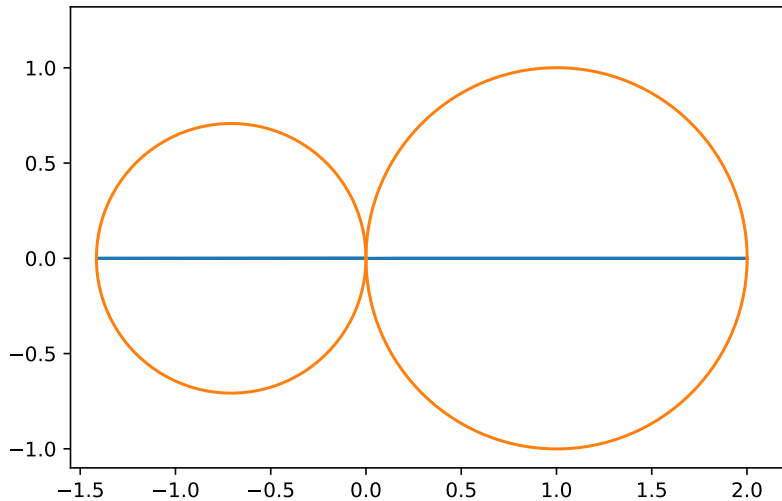
A few more facts about flowers:

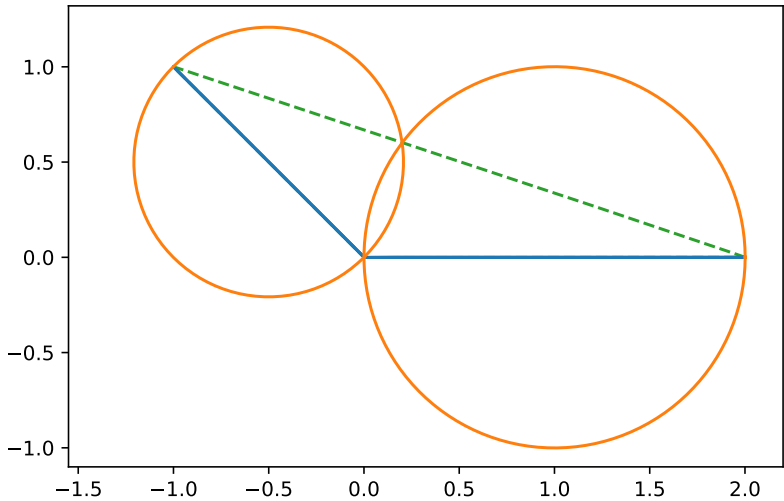
- ▶ If $F = \bigcup_{x \in A \subseteq \mathbb{R}^n} B_x$ is a flower, then necessarily $F^{\clubsuit} = \text{conv } A$ and this will be our construction of $F = K^{\clubsuit}$ from K :

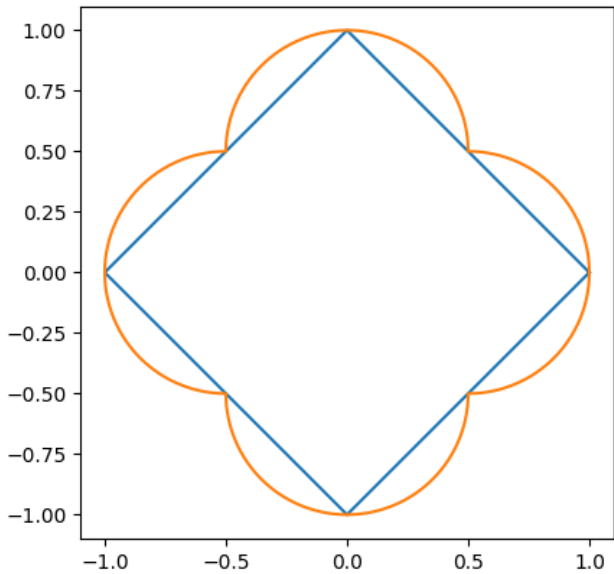
$$F = K^{\clubsuit} = \bigcup_{x \in K} B_x.$$

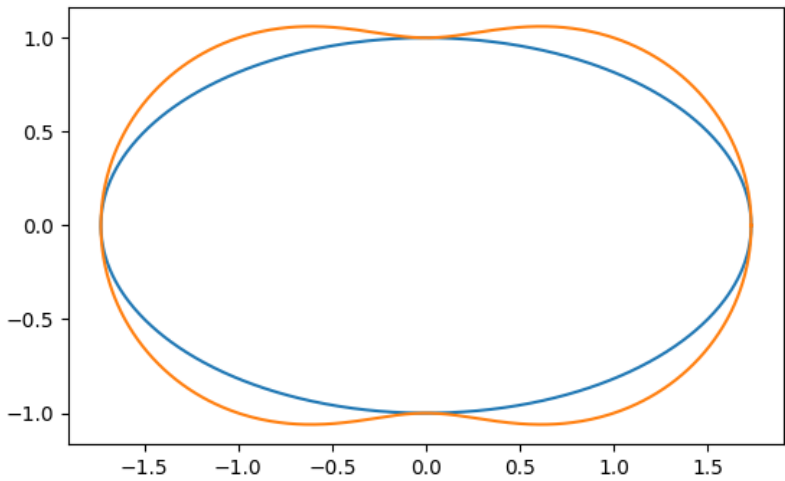
- ▶ $[0, x]^{\clubsuit} = B_x$, i.e. a single petal (Thales theorem).
- ▶ $(\text{conv}(K \cup T))^{\clubsuit} = K^{\clubsuit} \cup T^{\clubsuit}$ for all convex bodies K and T .











- ▶ If F_1 and F_2 are flowers, so are both the radial sum $F_1 \tilde{+} F_2$ (defined by $r_{F_1 \tilde{+} F_2} = r_{F_1} + r_{F_2}$) and the Minkowski sum $F_1 + F_2$. (Note: Minkowski sum of not-necessarily-convex sets!)
- ▶ If F is a flower and $E \subseteq \mathbb{R}^n$ is any linear subspace, then $F \cap E$ is also a flower, and in fact

$$P_E \left(F^{-\clubsuit} \right) = (F \cap E)^{-\clubsuit},$$

where P_E denotes the orthogonal projection onto E .

- ▶ Moreover $P_E F$ is also a flower, even though we do not have an independent description of $(P_E F)^{-\clubsuit}$. Since $P_E F \supseteq F \cap E$ we know that $(P_E F)^{-\clubsuit} \supseteq P_E \left(F^{-\clubsuit} \right)$, but we do not have a good understanding of this set. Perhaps, the notion of Sufficient enlargements, introduced by M. Ostrovskii, plays some role here.
- ▶ Finally, If F is a flower so is its convex hull $\text{conv } F$. In this case there is a description of $(\text{conv } F)^{-\clubsuit}$ in terms of $F^{-\clubsuit}$ and the so-called reciprocity map, but we will not explain it further here.

Example of use of “flower presentation” for convexity

For a function $f : S^{n-1} \rightarrow [0, \infty]$ its Alexandrov body is

$$A[f] = \left\{ x \in \mathbb{R}^n : \langle x, \theta \rangle \leq f(\theta) \text{ for all } \theta \in S^{n-1} \right\}.$$

Note that for $K \in \mathcal{K}_0$

$$A[h_K] = K$$

(for h_K is the support function of K).

We call $K' := A\left[\frac{1}{h_K}\right]$ a **reciprocal body**. Note that $K' \subseteq K^\circ$,

$K'' \supseteq K$, $K''' = K'$. Also, $K' = (K^\clubsuit)^\circ$.

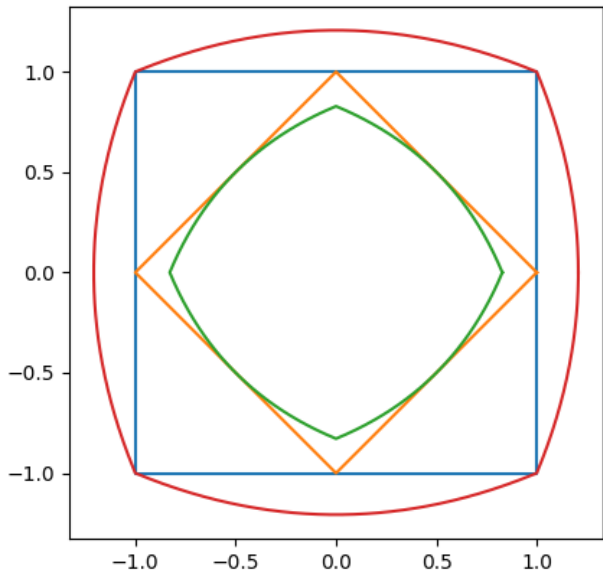
Question

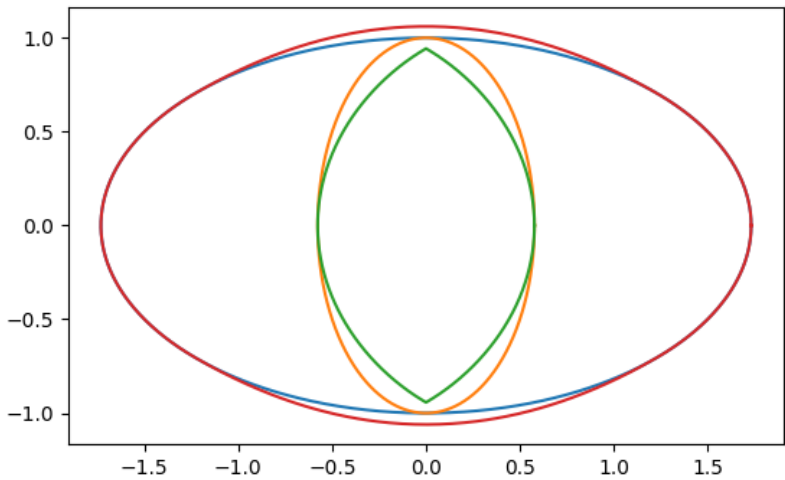
For which $K \in \mathcal{K}_0$ there exists $T \in \mathcal{K}_0$ such that

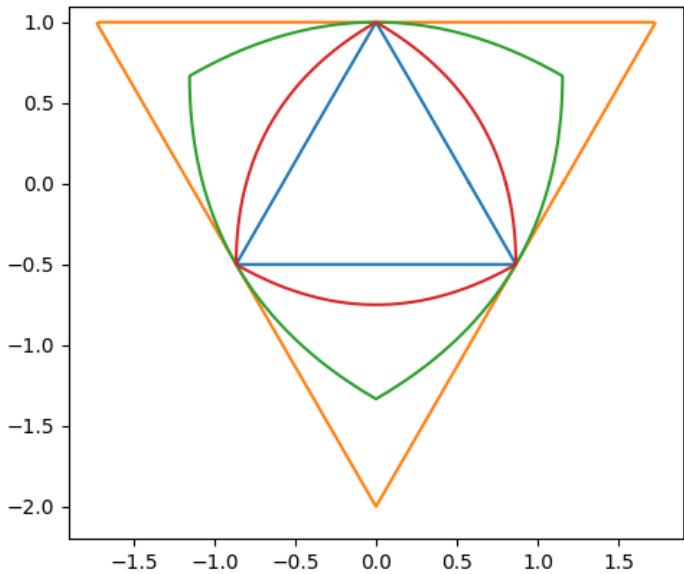
$$T' = A\left[\frac{1}{h_T}\right] = K?$$

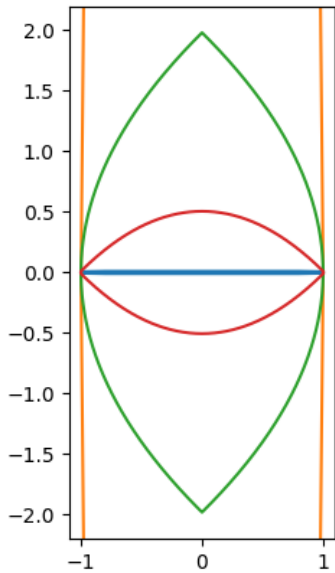
Answer (E. Milman, V. M., L. Rotem)

If and only if K^\clubsuit is convex!









Two facts that may be useful for our Asymptotic Geometric Analysis:

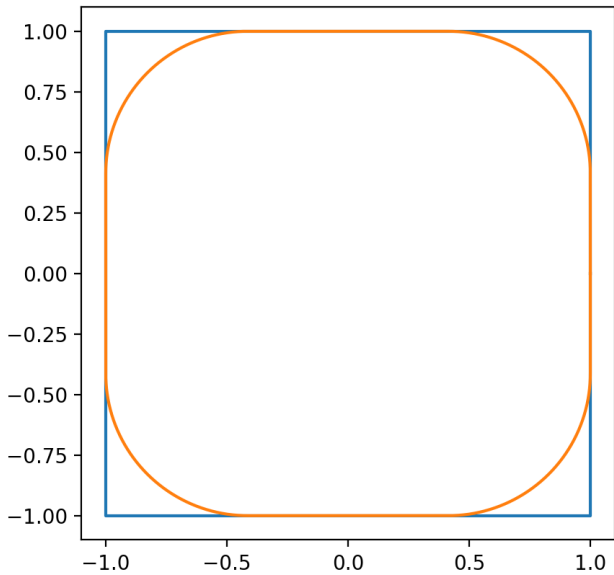
Fact 1

For every $K \in \mathcal{K}_0$ (with 0 in its interior) there exists a unique maximal flower $F \subseteq K$.

(It is convex; If $K = -K$ then $rB_2^n \subseteq F \subseteq 2rB_2^n$).

Fact 2

There also exists a unique minimal reciprocal body $T \supseteq K$.



There are other non-linear equations which are known to have convex solutions:

Theorem (I. Molchanov; no connection to flowers)

For every compact $K \supseteq B_2^n$ there exists a unique $Z \in \mathcal{K}_0$ such that

$$Z^\circ = Z + K.$$

Fact (A. Segal)

The same statement is true if the polar operation is changed to prime (i.e., take reciprocal body of Z):

$$Z' = Z \oplus K, \text{ if } K'' = K.$$

Note the summation also is changed to another one, which preserves reciprocity.

Next we discuss more non-linear constructions in convexity, based on flowers, like the power function K^λ , products and compositions of convex bodies.

Another statement about connection with AGA:

Let $r_F(\theta)$ be the radial function of a flower $F \in \mathcal{F}$ and fix $\epsilon > 0$.

Then there exists $N \leq c \cdot \frac{n}{\epsilon^2}$ rotations $\{u_i\}_{i=1}^N$ such that for some $r > 0$, and all $\theta \in S^{n-1}$,

$$(1 - \epsilon)r \leq \frac{1}{N} \sum_{i=1}^N r_{u_i F}(\theta) \leq (1 + \epsilon)r.$$

[I leave it to the listeners with background in AGA to find a correct reference as an exercise]

Think also on a partial case where F is a single petal.

Part 2

Consider a flower $F = \bigcup_{x \in A} B_x$, and set $K = F^{-\clubsuit} = \text{conv } A$. We call the representation $F = \bigcup_{x \in A} B_x$ is canonical if $A = \partial K$. Equivalently, for every $\theta \in S^{n-1}$ there exists a unique $x \in A$ of the form $x = r_\theta \theta$, $r_\theta \geq 0$.

Fix $f : [0, \infty) \rightarrow [0, \infty)$ such that $f(0) = 0$.

Definition

$f(F) := \bigcup B_{f(r_\theta)\theta}$ is a flower. For convex bodies we set $f(K) = f(K^{\clubsuit})^{-\clubsuit}$.

Examples

$$\begin{aligned} f(r) = r &\implies \text{co}\varphi(f(F)) = K^\circ \text{ where } K^{\clubsuit} = F, f(K) = K \\ f(r) = 1/r &\implies \text{co}\varphi(f(F)) = (K^\circ)', f(K)^\circ = (K^\circ)' \end{aligned}$$

Note $\text{co}\varphi(\bigcup B_{f(r_\theta)\theta}) = A[1/f(r_K)]$, so $f(K) = \left(A \left[\frac{1}{f(r_K(\theta))} \right] \right)^\circ$.

This is a naïve definition. However, it may also be useful for new geometric inequalities.

For example, take $f(x) = x^\lambda$ for some $0 < \lambda < 1$. Then

$$f(K) = ((1 - \lambda) \cdot B_2^n +_0 \lambda \cdot K^\circ)^\circ,$$

where $+_0$ is the *logarithmic mean* of Böröczky, Lutwak, Yang and Zhang.

Saroglou proved that in this case $|f(K)| \leq |B_2^n|^{1-\lambda} |K|^\lambda$.

The problem: If f_i , $i = 1, 2$, are two functions, then typically

$$(f_1 \circ f_2)(K) \neq f_1(f_2(K)).$$

(we do have $(f_1 \circ f_2)(K) \subseteq f_1(f_2(K))$ if f_1 is increasing)

We should correct this to build K^λ , $0 \leq \lambda \leq 1$, which satisfy the semigroup property, and any time we want to have

$$f_1 \circ f_2(K) = f_1(f_2(K)).$$

This is possible:

Theorem (V.M. - L. Rotem)

On the class of flowers there are maps $F \mapsto F^\lambda$, $0 \leq \lambda < \infty$, with the following properties:

1. $F^1 = F$ and $F^0 = B_2^n$.
2. If $F_1 \subseteq F_2$ then $F_1^\lambda \subseteq F_2^\lambda$.
3. $(tF)^\lambda = t^\lambda F^\lambda$ for $t \geq 0$.
4. F^λ is continuous with respect to both F and λ .
5. $(F^\mu)^\lambda = F^{\lambda\mu}$ for $0 < \lambda, \mu \leq 1$ and for $1 \leq \lambda, \mu \leq \infty$.
6. If $aB_2^n \subseteq F \subseteq \sqrt{2}aB_2^n$ for some $a > 0$ then $(F^\lambda)^{1/\lambda} = F$ for all $0 < \lambda \leq 1$.

And for the convex bodies from \mathcal{K}_0 we have built a power map s.t. the above conditions are satisfied.

There is a distinction between the cases $0 < \lambda < 1$ and $\lambda > 1$:

Proposition

If $K^{\lambda_1} = K^{\lambda_2}$ for $\lambda_1, \lambda_2 \leq 1$, $\lambda_1 \neq \lambda_2$, then $K = B_2^n$.

Example

Let $K = \left[-\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right]^n$ be the cube inscribed in S^{n-1} . Then $K^\lambda = K$ for all $\lambda \geq 1$.

For volumes, a result of Saroglou implies:

Theorem (Essentially Saroglou)

For every convex body K and $0 < \lambda < 1$ we have

$$|K^\lambda| \leq |B_2^n|^{1-\lambda} |K|^\lambda.$$

For $\lambda \geq 1$ the inequality is reversed (and easy).

Return to a general construction $f(K)$.

Now, more generally, consider a function $f(\theta, r) \geq 0$ for $\theta \in S^{n-1}$, $r \geq 0$.

Let $F = \bigcup B_{r(\theta)\theta}$. Then

$$f(F) := \bigcup_{\theta} B_{f(\theta, r(\theta))\theta}; \quad \text{Note } \text{co}\varphi(f(F)) = A[1/f(\theta, r(\theta))].$$

Similarly, for $K = F^{-\clubsuit}$, define

$$f(K) := f(F)^{-\clubsuit} = f(K^{\clubsuit})^{-\clubsuit}.$$

Then $f(K) = (A[1/f(\theta, r(\theta))])^\circ$

We use this to define **composition** of two convex bodies T and K .

First, for flowers

$$F_1 = \bigcup B_{r_1(\theta)\theta} \quad \text{and} \quad F_2 = \bigcup B_{r_2(\theta)\theta} \quad (\text{in canonical presentation}),$$

define $F_1 \circ F_2 := \rho_{F_1}(F_2)$ where ρ_{F_1} is a 1-homogeneous function built by the radial function $\rho_1(\theta)$ of F_1 , i.e. $\rho_{F_1}(\theta, r) = \rho_1(\theta) \cdot r$.

This means

$$F_1 \circ F_2 = \bigcup B_{\rho_1(\theta) \cdot r_2(\theta)\theta}.$$

Now let $F_1 = T^\clubsuit$ and $F_2 = K^\clubsuit$.

Then $r_2(\theta) = r_K(\theta)$, the radial function of K , and $\rho_1(\theta) = h_T(\theta)$.

So we define

$$T \circ K := \left(\bigcup B_{h_T(\theta) \cdot r_K(\theta)\theta} \right)^{\clubsuit}.$$

Note $T \circ T^\circ = B_2^n$ and $(T \circ K)^\circ = A\left[\frac{1}{h_T \cdot r_K}\right]$.

This may also be seen as

$$T \circ K = h_T(K) \equiv [h_T(K^\clubsuit)]^{-\clubsuit}$$

We have, connected with T , another function r_T , the radial function of T , and we may define a different composition

$$T \odot K := r_T(K) = [r_T(K^\clubsuit)]^{-\clubsuit}.$$

This is

$$\left[\bigcup B_{r_T(\theta) \cdot r_K(\theta)\theta} \right]^{-\clubsuit}.$$

So $T \odot K$ is a commutative “product”.

If $T = B_2^n$ then both compositions preserve K , i.e. the identical map on \mathcal{K}_0 .

Problem. Find bodies T s.t. the Brunn-Minkowski type inequality

$$|T \circ (K_1 + K_2)|^{1/n} \geq |T \circ K_1|^{1/n} + |T \circ K_2|^{1/n},$$

or equivalently

$$\left| T \circ \frac{K_1 + K_2}{2} \right| \geq \min \{ |T \circ K_1|, |T \circ K_2| \},$$

is correct (for any of 2 compositions h or r)?

Let us rewrite $T \odot K$ in an explicit form.

Define $T \cdot K$ to be the star body with the radial function

$$r_{T \cdot K}(\theta) = r_T(\theta) \cdot r_K(\theta).$$

Then $T \odot K = \text{conv}(T \cdot K)$.

In the same notation we may use $A \cdot B$ for flowers.

Then

$$T \circ K = \text{conv}(T^{\clubsuit} \cdot K),$$

because the radial function of T^{\clubsuit} is $h_T(\theta)$.

Part 3: Dvoretzky Theorem for Flowers

For this part you only need to remember the following facts about flowers:

- ▶ A flower is a union of balls: $F = \bigcup_{i \in I} B_i$. One may assume either that $0 \in B_i$ or the stronger assumption that $0 \in \partial B_i$ and the definitions are equivalent.
- ▶ There is a bijection between convex bodies and flowers:

$$K \leftrightarrow K^{\clubsuit} = \bigcup_{x \in \partial K} B\left(\frac{x}{2}, \frac{|x|}{2}\right).$$

- ▶ Under this bijection we have $K^{\clubsuit} \cap E = (P_E K)^{\clubsuit}$ for every subspace $E \subseteq \mathbb{R}^n$.

For every star body $A \subseteq \mathbb{R}^n$ its distance to Euclidean is

$$d(A, B_2^n) = \inf \left\{ \frac{R}{r} : rB_2^n \subseteq A \subseteq R \cdot B_2^n \right\}.$$

It is easy to check that $d(K, B_2^n) = d(K^{\clubsuit}, B_2^n)$.

Theorem (Dvoretzky (with Milman's estimate))

For every symmetric convex body $K \subseteq \mathbb{R}^n$ (i.e. $K = -K$) and every $\epsilon > 0$ there exists a subspace $E \subseteq \mathbb{R}^n$ of dimension $\dim E \geq C(\epsilon) \log n$ such that

$$d(K \cap E, B_2^n) \leq 1 + \epsilon. \quad (\odot)$$

- ▶ $C(\epsilon)$ is some function that depends only on $\epsilon > 0$. We will not discuss the best dependence on ϵ .
- ▶ The bound $\dim E \geq c \cdot \log n$ is optimal, as can be computed for $K = [-1, 1]^n$.
- ▶ Instead of (\odot) one may demand that

$$d(P_E K, B_2^n) \leq 1 + \epsilon.$$

The two versions are the same by duality.

The following is a restatement of the same theorem:

Corollary

For every symmetric flower $F \subseteq \mathbb{R}^n$ and every $\epsilon > 0$ there exists a subspace $E \subseteq \mathbb{R}^n$ of dimension $\dim E \geq C(\epsilon) \log n$ such that

$$d(F \cap E, B_2^n) \leq 1 + \epsilon.$$

Again, $\log n$ is optimal.

Surprisingly, something much better happens for projections:

Theorem (V. Milman - L. R)

For every symmetric flower $F \subseteq \mathbb{R}^n$ and every $\epsilon > 0$ there exists a subspace $F \subseteq \mathbb{R}^n$ of dimension $\dim F \geq C(\epsilon)n$ such that

$$d(P_E F, B_2^n) \leq 1 + \epsilon.$$

This is not a reformulation of some known theorem about convex bodies. We will now sketch a proof.

Step 1

F is flower, but in general its only a star body, not a convex body.

Claim

If $F \subseteq \mathbb{R}^n$ is a flower, so is $\text{conv } F$

Proof (During the talk):

Step 2

$\text{conv } F$ is both a flower and convex. Such sets are very good:

Claim

If $F \subseteq \mathbb{R}^n$ is an origin symmetric convex flower, then $d(F, B_2^n) \leq 2$.

Proof (During the talk):

Interlude

Milman's version of Dvoretzky's theorem actually gives more than previously stated:

Theorem (Milman)

For every symmetric convex body $K \subseteq \mathbb{R}^n$ (i.e. $K = -K$) and every $\epsilon > 0$ there exists a subspace $E \subseteq \mathbb{R}^n$ of dimension $\dim E \geq C(\epsilon) \cdot n \cdot \left(\frac{w(K)}{R(K)}\right)^2$ such that

$$d(P_E K, B_2^n) \leq 1 + \epsilon.$$

Here $w(K)$ is the mean width of K and $R(K)$ is the outer radius of K . Obviously $\frac{w(K)}{R(K)} \geq d(K, B_2^n)$.

Plugging in $K = \text{conv } F$ we obtain:

Corollary

If $F \subseteq \mathbb{R}^n$ is an origin symmetric flower then for every $\epsilon > 0$ there exists a subspace $E \subseteq \mathbb{R}^n$ of dimension $\dim E \geq C(\epsilon)n$ such that

$$d(\text{conv}(P_E F), B_2^n) = d(P_E(\text{conv } F), B_2^n) \leq 1 + \epsilon.$$

But we want $d(P_E F, B_2^n)$, not $d(\text{conv}(P_E F), B_2^n)$. For general star bodies there is no hope to bound the first using the second. But $P_E F$ is not a general star body!

Step 3 (Easy)

Claim

If $F \subseteq \mathbb{R}^n$ is a flower then so is $P_E F$.

Proof.

If $F = \bigcup_i B_i$ and $0 \in B_i$, then

$$P_E F = \bigcup_i P_E B_i,$$

and every $P_E B_i$ is a ball containing 0. □

The crucial final step, and the main part of the proof, is a stability estimate for flowers:

Step 4

Proposition

Let $F \subseteq \mathbb{R}^n$ be a flower, and assume that $d(\text{conv } F, B_2^n) \leq 1 + \epsilon$ for some $\epsilon < \frac{1}{10}$. Then $d(F, B_2^n) \leq 1 + 3\sqrt{\epsilon}$.

Proof Sketch (During the talk):