On the L^p Aleksandrov problem for negative p

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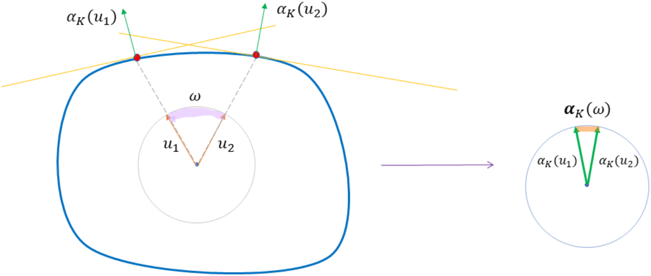
Integral Curvature

- The integral curvature of $K \in \mathcal{K}_o^n$:

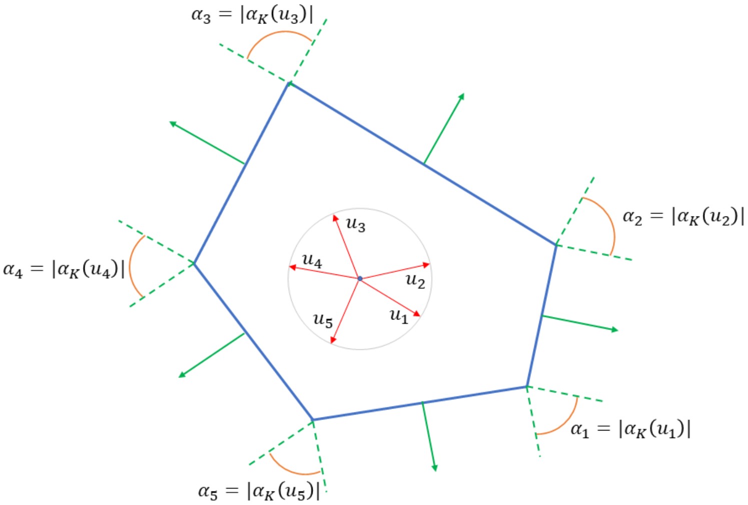
$$J(K,\omega) = \mathcal{H}^{n-1}(\boldsymbol{\alpha}_K(\omega))$$

for every Borel $\omega \subset S^{n-1}$ (Aleksandrov 1942)

  - Radial Gauss map $\boldsymbol{\alpha}_K(\omega)$ maps radial vectors to normal vectors
  - Measure of the normal cone of the radial projection to ∂K



Integral Curvature for a Polygon



$$J(P,\cdot) = \alpha_1\delta_{u_1} + \alpha_2\delta_{u_2} + \alpha_3\delta_{u_3} + \alpha_4\delta_{u_4} + \alpha_5\delta_{u_5}$$

Classical Aleksandrov Problem

Problem (Aleksandrov 1942)

What are the necessary and sufficient conditions on a Borel measure μ on S^{n-1} so that

$$J(K, \cdot) = \mu$$

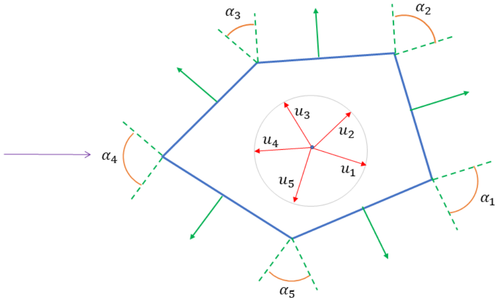
for some $K \in \mathcal{K}_o^n$?

- Classical Aleksandrov problem is a type of Minkowski problem
  - Contrast with classical Minkowski problem:

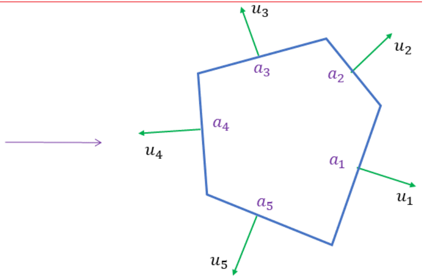
$$S_K(\cdot) = \mu$$

Classical Aleksandrov Problem vs Minkowski Problem

$$\mu(u) = \begin{cases} \alpha_1, & \text{for } u = u_1 \\ \alpha_2, & \text{for } u = u_2 \\ \alpha_3, & \text{for } u = u_3 \\ \alpha_4, & \text{for } u = u_4 \\ \alpha_5, & \text{for } u = u_5 \end{cases}$$



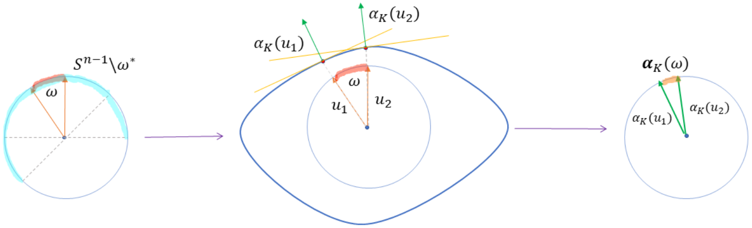
$$\mu(u) = \begin{cases} a_1, & \text{for } u = u_1 \\ a_2, & \text{for } u = u_2 \\ a_3, & \text{for } u = u_3 \\ a_4, & \text{for } u = u_4 \\ a_5, & \text{for } u = u_5 \end{cases}$$



Aleksandrov Condition

- A Borel measure μ on S^{n-1} satisfies the Aleksandrov condition if
$$\mu(\omega) < \mathcal{H}^{n-1}(S^{n-1} \backslash \omega^*)$$
for each convex $\omega \subset S^{n-1}$, where
$\omega^* = \left\{ v \in S^{n-1} : \; v \cdot u \leq 0 \; \forall u \in \omega \right\}.$
- The integral curvature measure $J(K, \cdot)$ for $K \in \mathcal{K}_o^n$ satisfies the Aleksandrov condition

$$\boldsymbol{\alpha}_K(\omega) \subset S^{n-1} \backslash \omega^*$$



The Classical Aleksandrov Problem Solution

Theorem (Aleksandrov 1942)

Suppose μ is a finite Borel measure on S^{n-1}. Then $\mu = J(K, \cdot)$ for some $K \in \mathcal{K}_o^n$ if and only if $|\mu| = o_n$ and

$$\mu(\omega) < \mathcal{H}^{n-1}(S^{n-1} \backslash \omega^*) \tag{0.1}$$

for each convex $\omega \subset S^{n-1}$, where $\omega^* = \left\{ v \in S^{n-1} : \ v \cdot u \leq 0 \ \forall u \in \omega \right\}$. Furthermore, K is unique up to scaling.

- (Firey 1962) For every $p \geq 1$, $K, L \in \mathcal{K}_o^n$, and $a, b \geq 0$, define

$$h_{aK +_p bL} = \left(a \cdot h_K^p + b \cdot h_L^p\right)^{\frac{1}{p}}$$

- Generalized $\forall p \in \mathbb{R}$,

$$a \cdot K +_p b \cdot L = \left[\left(a \cdot h_K^p + b \cdot h_L^p\right)^{\frac{1}{p}}\right]$$

- Actively researched when (Lutwak 1993) discovered the concept of the L^p surface area measure
  - For each $K, L \in \mathcal{K}_o^n$, defined by variational formula

$$\frac{d}{dt} V(K +_p t \cdot L)\bigg|_{t=0} = \frac{1}{p} \int_{S^{n-1}} h_L(u)^p \, dS_p(K, u)$$

Problem

For all $p \in \mathbb{R}$, what are the necessary and sufficient conditions on a given Borel measure μ on S^{n-1} so that there exists a $K \in \mathcal{K}_o^n$ with

$$\mu = S_p(K, \cdot) = h_K(\cdot)^{1-p} dS(K, \cdot)?$$

- $p = 1$ case is the classical Minkowski problem
- $p = 0$ case is the logarithmic Minkowski problem
- $p = -n$ case (largely unsolved) is the centro-affine Minkowski problem

L^p Brunn-Minkowski Fields of Research

- **L^p Minkowski Problem:** Lutwak, Chou, Wang, Böröczky, LYZ, Stancu, Zhu, Chen, Hug, Li, Jian, Lu, Haberl, Huang, Liu, Kolesnikov, Milman, Oliker, Trinh, Bianchi, Colesanti, Xing, Xiong, Zou, Gardner, Ye, Weil, Hu, Ma, Shen, Xi, Leng, ...
- **L^p Affine Surface Area and Valuations:** Lutwak, Ludwig, Paouris, Werner, Ye, Meyer, Schütt, Haberl, Parapatits, Reitzner, Colesanti, Hug, Wannerer, ...
- **L^p Affine Isoperimetric Inequalities:** LYZ, Haberl, Schuster, Milman, Yehudayoff, Wang, Liu, Werner, Colesanti, Campi, Gronchi, ...
- **L^p Affine Sobolev Inequalities:** Cianchi, LYZ, Haberl, Schuster, Kneifacz, Haddad, Jiménez, Montenegro, Napoli, Nguyen, Wang, Xiao, Zhai, ...

L^p Integral Curvature

- $p \in \mathbb{R}$ and $a, b \geq 0$, define L^p harmonic combination

$$a \cdot K \hat{+}_p b \cdot L = (a \cdot K^* +_p b \cdot L^*)^*$$

- (Huang-LYZ 2018, JDG) defined the L^p integral curvature by variational formula for each $K, L \in \mathcal{K}_o^n$:

$$\frac{d}{dt}\mathcal{E}(K \hat{+}_p t \cdot L)\Big|_{t=0} = \begin{cases} \frac{1}{p}\int_{S^{n-1}} \rho_L(u)^{-p} \, dJ_p(K,u) & , \text{ for } p \neq 0 \\ -\int_{S^{n-1}} \log(\rho_L(u)) \, dJ(K,u) & , \text{ for } p = 0 \end{cases}$$

where the entropy is

$$\mathcal{E}(K) = -\int_{S^{n-1}} \log h_K(v) \, dv$$

- Relationship to classical integral curvature

$$dJ_p(K, \cdot) = \rho_K^p \, dJ(K, \cdot)$$

- $J_p(K,\cdot)$ defined by a variation on entropy $\mathcal{E}$ of the L^p harmonic combination
    - $S_p(K,\cdot)$ defined by a variation on volume V of the L^p Minkowski sum
- $J_p(K,\cdot)$ is a function on the radial sphere
    - $S_p(K,\cdot)$ is a function on the normal sphere
- $dJ_p(K,\cdot) = \rho_K^p \, dJ(K,\cdot)$
    - $dS_p(K,\cdot) = h_K(\cdot)^{1-p} dS(K,\cdot)$
- $J(K,\cdot)$ measures exterior angles
    - $S(K,\cdot)$ measures surface area

L^p Aleksandrov Problem

Problem

Fix $p \in \mathbb{R}$. What are the necessary and sufficient conditions on a given Borel measure μ on S^{n-1} so that there exists a convex body $K \in \mathcal{K}_o^n$ with

$$J_p(K, \cdot) = \mu \ ?$$

- If μ has density f, equivalent to PDE

$$\det\left(\nabla_{ij}^2 h + h\delta_{ij}\right) = \frac{\left(|\nabla h|^2 + h^2\right)^{\frac{n}{2}}}{h^{1-p}} f$$

L^p Aleksandrov Problem Results

- (Huang-LYZ 2018) completely solved existence for $p > 0$
- (Huang-LYZ 2018) solved existence under some strong conditions when $p < 0$
  - Measure is even and vanishes on all great subspheres
  - Excludes many shapes, including polytopes
- (Zhao 2019, Proc. AMS) addressed this polytope gap
  - $-1 < p < 0$
  - Measure is even and discrete

Recent Progress for $p < 0$ Case (M. 2021)

- Completely solve the symmetric case for $-1 < p < 0$

Theorem

μ is even and $-1 < p < 0$. Then $\exists K \in \mathcal{K}_e^n$ s.t. $J_p(K, \cdot) = \mu$ iff μ is not completely concentrated on lower dimensional subspace.

- A sufficient measure concentration condition for the symmetric case and $p \leq -1$

Theorem

$p \leq -1$, μ is even and satisfies

$$\frac{\mu(\xi)}{\mu(S^{n-1})} \leq C(n)^p$$

for all great subspheres $\xi \subset S^{n-1}$, where $C(n) = \exp\left[\frac{1}{2}\left(\psi\left(\frac{n}{2}\right) - \psi\left(\frac{1}{2}\right)\right)\right]$. Then $\exists K \in \mathcal{K}_e^n$ s.t. $J_p(K, \cdot) = \mu$.

Outline for Proofs

- Convert to maximization problem
- Compactness of maximizing sequence
  - Blaschke's Selection $\implies$ convergence to a compact convex Q^0
- Non-collapse for Q^0
- $o \in \operatorname{int} Q^0$
  - Previous two points are equivalent for o-symmetric case
  - Most difficult part of proofs for both theorems

- Denote o_n to be the surface area of the unit sphere. For any nonzero, finite Borel measure μ on S^{n-1} and $p \neq 0$, we consider the following functionals
  - $\widetilde{\Phi}_p(Q) = \exp\left(\frac{1}{o_n}\mathcal{E}(Q)\right) \cdot \left(\int_{S^{n-1}} \rho_Q^{-p}(u)\, d\mu(u)\right)^{-\frac{1}{p}}$
  - $\Phi_p(Q) = -\frac{1}{p}\log\left(\int_{S^{n-1}} \rho_Q^{-p}(v)\, d\mu(v)\right) + \frac{1}{o_n}\mathcal{E}(Q)$

Lemma

Suppose $p \neq 0$ and μ is an even Borel measure on S^{n-1}. If $K \in \mathcal{K}_e^n$ satisfies

$$o_n = \int_{S^{n-1}} \rho_K^{-p}(u)\, d\mu(u) \tag{0.2}$$

and $\widetilde{\Phi}_p(K) = \sup\left\{\widetilde{\Phi}_p(Q) : \ Q \in \mathcal{K}_e^n\right\}$ or $\Phi_p(K) = \sup\{\Phi_p(Q) : \ Q \in \mathcal{K}_e^n\}$, then $\mu = J_p(K, \cdot)$.

Sketch of Proof $p \in (-1, 0)$

- Maximize

$$\widetilde{\Phi}_p(Q) = \exp\left(\frac{-1}{o_n}\int_{S^{n-1}} \log h_Q(v)\ dv\right)\cdot\left(\int_{S^{n-1}} \rho_Q^{-p}(u)\ d\mu(u)\right)^{-\frac{1}{p}}$$

- Show existence of maximizer Q^0
- Show that Q^0 is nondegenerate (i.e. $o \in \operatorname{int} Q^0$)
- Consider a cylindrical thickening given by $K^t = Q^0 + tB^{n-k}$

Sketch of Proof $p \in (-1, 0)$

- (Contradiction Approach) Assume optimizer Q^0 spans a $k < n$ dimensional subspace
- Contradiction by showing $\widetilde{\Phi}_p(K^t) > \widetilde{\Phi}_p(Q^0)$ for small $t > 0$
  - $\frac{\widetilde{\Phi}_p(K^t)}{\widetilde{\Phi}_p(Q^0)} \to 1$
  - $\frac{d}{dt}\frac{\widetilde{\Phi}_p(K^t)}{\widetilde{\Phi}_p(Q^0)} > 0$

Delicate Estimates

- Estimate for $\Delta_2(Q^0,t) := \dfrac{\left(\int_{S^{n-1}} \rho_{K^t}^{-p}(u)\, d\mu(u)\right)^{-\frac{1}{p}}}{\left(\int_{S^{n-1}} \rho_{Q^0}^{-p}(u)\, d\mu(u)\right)^{-\frac{1}{p}}}$
  - $\lim_{t\to 0^+} \tilde{\Delta}_2(Q^0,t) = 1$
  - $\lim_{t\to 0^+} \frac{d}{dt}\tilde{\Delta}_2(Q^0,t) \sim t^{-p-1}$
- Estimate for $\Delta_1(Q^0,t) := \dfrac{\exp\left(\frac{-1}{\omega_n}\int_{S^{n-1}} \log h_{K^t}(v)\ dv\right)}{\exp\left(\frac{-1}{\omega_n}\int_{S^{n-1}} \log h_{Q^0}(v)\ dv\right)}$
  - $\lim_{t\to 0^+} \tilde{\Delta}_1(Q^0,t) = 1$
  - $\lim_{t\to 0^+} \frac{d}{dt}\tilde{\Delta}_1(Q^0,t) \gtrsim \log(t)$.
    - Proved first for the case $Q^0 = rB^k$ and generalized to arbitrary k-dimensional symmetric convex bodies
- $\frac{d}{dt}\frac{\widetilde{\Phi}_p(K^t)}{\widetilde{\Phi}_p(Q^0)} \sim t^{-p-1} + \log(t)$

Sketch of Proof $p \leq -1$

- Maximize

$$\Phi_p(Q) = -\frac{1}{p}\log\left(\int_{S^{n-1}} \rho_Q^{-p}(u)\, d\mu(u)\right) - \frac{1}{o_n}\int_{S^{n-1}} \log h_Q(v)\ dv$$

- Suppose $\{Q_l\}$ is maximizing sequence
- Φ_p is scale invariant $\implies$ rescale

$$\left(\int_{S^{n-1}} \log\left(h_{Q_l}(u)\right)\ du\right) = 0$$

- $Q_l \subset MB^n$, for some $M > 0 \implies \{Q_l\} \to Q^0$
- Prove Q_0 is nondegenerate by contradiction
- Assume $\exists u_0 \in S^{n-1}$ such that $h_{Q_0}(\pm u_0) = 0$

- $\forall \delta > 0$, define $\omega_\delta(u_0) := \left\{ v \in S^{n-1} : \ |v \cdot u_0| > \delta \right\}$

$$\begin{aligned}\Phi_p(Q_l) &= -\frac{1}{p}\log\left(\int_{S^{n-1}} \rho_Q^{-p}(v)\, d\mu(v)\right)\\ &\leq -\frac{1}{p}\log\left(\left(\sup_{\omega_\delta} \rho_{Q_l}^{-p}(v) - M^{-p}\right)\mu(\omega_\delta) + M^{-p}\mu(S^{n-1})\right)\end{aligned}$$

- Taking limits and applying the measure concentration, showed:

$$\begin{aligned}\lim_{l\to\infty} \Phi_p(Q_l) &\leq -\frac{1}{p}\log(\mu(S^{n-1}))\\ &= \Phi_p(B^n).\end{aligned}$$

contradicts assumption that Q_0 is maximizer

Calculation of $C(n)$

$$\begin{aligned}
C(n) &= \exp\left(\frac{-1}{o_n}\int_{S^{n-1}} \log|v_0 \cdot u| \; du\right) \\
&= \exp\left(-\frac{2 \cdot o_{n-1}}{o_n}\int_0^{\frac{\pi}{2}} \left(\sin^{n-2}\phi\right)\log(\cos\phi)\; d\phi\right) \\
&= \lim_{q\to 0} \exp\left[\frac{1}{q}\log\left(\frac{2\cdot o_{n-1}}{o_n}\int_0^{\frac{\pi}{2}}\left(\sin^{n-2}\phi\right)\left(\cos^{-q}\phi\right)\; d\phi\right)\right] \\
&= \exp\left[\lim_{q\to 0}\frac{1}{q}\log\left(\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}\cdot\frac{\Gamma\left(\frac{1-q}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-q}{2}\right)}\right)\right] \\
&= \exp\left[\frac{1}{2}\left(\psi\left(\frac{n}{2}\right) - \psi\left(\frac{1}{2}\right)\right)\right]
\end{aligned}$$

- Note that $C(n)^p \to 0$ as $n \to \infty$ at rate $O(n^{\frac{p}{2}})$

Open Problems

- Optimal measure concentration condition for existence in L^p Aleksandrov for $p \leq -1$
- Eliminating the origin-symmetry assumption
- Analogous unsolved questions for L^p Dual Minkowski problem
- L^p Dual Minkowski problem
  - $p < 0$ and $q > 0$
  - $p < 0$ and $q < 0$

End

Thanks for listening!