Colorful and Fractional Quantitative Helly Theorems

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Part I.
Geometric Helly-type Theorems:
Intersection patterns of convex sets in $\mathbb{R}^d$

Bárány, Katchalski, Pach '82

Quantitative Volume Theorem [BKP'82]

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ such that any 2d of them have intersection of volume at least 1.
Then $\cap\mathcal{F}$ is of volume at least d^{-2d^2}.

- 2d cannot be improved — Rough approximation.
- N. '16 Volume bound improved to $C^d d^{-2d}$.
- Brazitikos '17 the same method yields $C^d d^{-3d/2}$.
- Ivanov, N. Functional version – Talk by G.I. on Sept. 12.

Does approximating the volume so poorly (up to a factor d^{cd}) make sense?

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Bárány, Füredi '87: Computing the volume is difficult

No deterministic polynomial algorithm approximates the volume better than up to a factor $\left(\frac{d}{\log d}\right)^d$.

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Helly-type theorems

$\mathcal{C}, \mathcal{C}_j$: finite families of convex sets in $\mathbb{R}^d$.

Classical	Quantitative
Helly 1913: If $\bigcap \mathcal{C} = \emptyset$, then $\exists C_1, \ldots C_{d+1} \in \mathcal{C} : \bigcap C_i = \emptyset$.	**Quantitative Volume Thm [Bárány, Katchalski, Pach '82]**
Colorful Helly Thm. [Lovász '74, Bárány '82]: If $\bigcap \mathcal{C}_j = \emptyset$ for all $j \in [d+1]$, then $\exists$ colorful choice $C_1 \in \mathcal{C}_1, \ldots, C_{d+1} \in \mathcal{C}_{d+1} : \bigcap C_j = \emptyset$.	**[Damásdi, Földvári, N.]**, **[Sarkar, Xue, Soberón]**
Fractional Helly Thm. [Katchalski, Liu '79]: Let $\alpha \in (0,1)$. Assume $\alpha\binom{\|\mathcal{C}\|}{d+1}$ of the $(d+1)$-tuples have non-empty intersection. Then $\exists$ clique of size $\beta\|\mathcal{C}\|$.	**[Jung, N.]**

Inscribed Ellipsoid /Fine approximation: $n \approx d^2$/

Behrend '37

$\mathcal{C}$ – a family of convex bodies in $\mathbb{R}^2$. Assume that the intersection of any 5 members contains an ellipse of area 1.
Then $\bigcap \mathcal{C}$ contains an ellipse of area 1.

Remark by Danzer, Grünbaum, Klee '63

Same holds in $\mathbb{R}^d$ with $d(d+3)/2$.

- d(d+3)/2 cannot be improved.
- By John's theorem,

volume $\approx$ volume of inscribed ellipsoid.

Inscribed Ellipsoid – Colorful /Fine approx.: $n \approx d^2$/

Lovász '74, Bárány '82: Colorful Helly Theorem

$\mathcal{C}_1, \ldots, \mathcal{C}_{d+1}$ – finite families of convex bodies in $\mathbb{R}^d$. Assume that for any colorful choice of $d+1$ sets, $C_i \in \mathcal{C}_i$ for each $i \in [d+1]$, the intersection $\bigcap_{i=1}^{d+1} C_i$ is not empty.

Then, there is an $i \in [d+1]$ such that $\bigcap_{C \in \mathcal{C}_i} C$ is not empty.

Inscribed Ellipsoid – Colorful /Fine approx.: $n \approx d^2$/

Damásdi, Földvári, N.: Colorful generalization of Behrend's result

$\mathcal{C}_1, \ldots, \mathcal{C}_{\frac{d(d+3)}{2}}$ – finite families of convex bodies in $\mathbb{R}^d$. Assume that for any colorful choice of $\frac{d(d+3)}{2}$ sets, $C_i \in \mathcal{C}_i$ for each $i \in \left[\frac{d(d+3)}{2}\right]$, the intersection $\bigcap_{i=1}^{\frac{d(d+3)}{2}} C_i$ contains an ellipsoid of volume 1.

Then, there is an $i \in \left[\frac{d(d+3)}{2}\right]$ such that $\bigcap_{C \in \mathcal{C}_i} C$ contains an ellipsoid of volume 1.

Our main colorful result

Lovász '74, Bárány '82: Colorful Helly Theorem

$\mathcal{C}_1, \ldots, \mathcal{C}_{d+1}$ – finite families of convex bodies in $\mathbb{R}^d$. Assume that for any colorful choice of $d+1$ sets, $C_i \in \mathcal{C}_i$ for each $i \in [d+1]$, the intersection $\bigcap_{i=1}^{d+1} C_i$ is not empty.

Then, there is an $i \in [d+1]$ such that $\bigcap_{C \in \mathcal{C}_i} C$ is not empty.

Note: Fewer color classes $\longrightarrow$ Stronger theorem.

Our main colorful result

Damásdi, Földvári, N.: Few color classes

$\mathcal{C}_1, \ldots, \mathcal{C}_{3d}$ – finite families of convex bodies in $\mathbb{R}^d$. Assume that for any colorful choice of $2d$ sets, $C_{i_k} \in \mathcal{C}_{i_k}$ for each $k \in [2d]$ with $1 \leq i_1 < \ldots < i_{2d} \leq 3d$, the intersection $\bigcap_{k=1}^{2d} C_{i_k}$ contains an ellipsoid of volume 1.

Then, there exists an $i \in [3d]$ such that $\bigcap_{C \in \mathcal{C}_i} C$ contains an ellipsoid of volume $d^{-O(d^2)}$.

Open:

- $2d$ in place of $3d$ should hold.
- $d^{-O(d)}$ in place of $d^{-O(d^2)}$?

Our main fractional result

Katchalski-Liu '79: Fractional Helly Theorem

For every $d \geq 1$ and $\alpha \in (0,1)$, there is a $\beta \in (0,1)$ s.t.:

$\mathcal{C}$ – a finite family of convex bodies in $\mathbb{R}^d$.

Assume that among all subfamilies of $\mathcal{C}$ of size $d+1$, there are at least $\alpha\binom{|\mathcal{C}|}{d+1}$ for whom the intersection of the $d+1$ members is non-empty.

Then, there is a subfamily $\mathcal{C}' \subset \mathcal{C}$ with $|\mathcal{C}'| \geq \beta|\mathcal{C}|$ such that $\bigcap \mathcal{C}'$ is non-empty.

Our main fractional result

Jung, N.: Quantitative Fractional Helly Theorem

For every $d \geq 1$ and $\alpha \in (0,1)$, there is a $\beta \in (0,1)$ s.t.:

$\mathcal{C}$ – a finite family of convex bodies in $\mathbb{R}^d$.

Assume that among all subfamilies of $\mathcal{C}$ of size $3d+1$, there are at least $\alpha\binom{|\mathcal{C}|}{3d+1}$ for whom the intersection of the $3d+1$ members contains an ellipsoid of volume 1.

Then, there is a subfamily $\mathcal{C}' \subset \mathcal{C}$ with $|\mathcal{C}'| \geq \beta|\mathcal{C}|$ such that $\bigcap \mathcal{C}'$ contains an ellipsoid of volume d^{-cd^2}.

Open:

- $2d$ in place of $3d+1$ should hold.
- d^{-cd} in place of d^{-cd^2}?

Part II.
Combinatorial Helly-type Theorems:
Intersection patterns of abstract set systems

Abstract Helly Type Theorems

A *hypergraph* on the ground set X is any family $H \subset 2^X$.

Eg.: $X = \mathrm{Cvx}(d) = \{\text{convex sets in } \mathbb{R}^d\}$.

$$H = \left\{ \{C_1, \ldots, C_{d+1}\} \; : \; \bigcap_{i \in [d+1]} C_i \neq \emptyset \right\},$$

a $(d+1)$-uniform hypergraph.

$Y \subset X$ is a *clique* of an $(d+1)$-uniform hypergraph, if $\binom{Y}{d+1} \subseteq H$.
By Helly's Thm, the intersection of all members of a clique is non-empty.

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Restate the Fractional Helly Theorem:
For any d, there is a $\beta : (0,1) \to (0,1)$ such that if $X = \mathrm{Cvx}(d)$, and H is as above, and $Y \subseteq X$ is finite such that $|H|_Y| \geq \alpha \binom{|Y|}{d+1}$, then there is a $Z \subseteq Y$ with $|Z| \geq \beta(\alpha)|Y|$ such that Z is a clique of H.

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Restate the Colorful Helly Theorem:

Let $Y \subset \mathrm{Cvx}(d)$, and $Y = \uplus_{i=1}^{r} Y_i$ be a *partition* of Y. If every colorful selection $s \in Y_1 \times \cdots \times Y_{d+1}$ is in H then there is an $i \in [d+1]$ such that Y_i is a clique.

Abstract Helly Type Theorems

$$\left.\begin{array}{l}\text{Colorful Helly Theorem,}\\ \text{Fractional Helly Theorem,}\\ \text{(p,q) Theorem [Alon, Kleitman '92]}\\ \ldots\end{array}\right\} \rightarrow \text{ Properties of hypergraphs.}$$

Alon, Kalai, Matoušek, Meshulam '02:

Fractional Helly Property $\Longrightarrow$ (p, q) Property.

Holmsen '19:

Colorful Helly Property $\Longrightarrow$ Fractional Helly Property.

Abstract Helly Type Theorems

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How to phrase a quantitative abstract theorem?

The assumption and the conclusion involve two distinct hypergraphs.

Quantitative Abstract Helly Type Theorems

Definition (Jung, N.)

$r_1, r_2 \in \mathbb{Z}^+$, X a set.

The hypergraphs $H_1, H_2 \subseteq 2^X$ are a *Colorful Helly-type Hypergraph Pair:*

For any subset $Y \subseteq X$ and partition $Y = \cup_{i=1}^{r_1} Y_i$, if $Y_1 \times \ldots \times Y_{r_1} \subseteq H_1$, then there is an $i \in [r_1]$ such that $\binom{Y_i}{r_2} \subseteq H_2$.

Fractional Helly-type Hypergraph Pair: Similar definition.

Quantitative Abstract Helly Type Theorems

Restatement of Quantitative Colorful Helly Theorem

$X = \mathrm{Cvx}(d)$.

$$H_1 = \left\{ \{C_1, \ldots, C_{3d}\} \ : \ \bigcap_{i \in [3d]} C_i \text{ contains an ellipsoid of vol } 1 \right\},$$

a $(3d)$-uniform hypergraph.

$$H_2 = \left\{ \{C_1, \ldots, C_{2d}\} \ : \ \bigcap_{i \in [2d]} C_i \text{ contains an ellipsoid of vol } c(d) \right\}$$

a $(2d)$-uniform hypergraph.

Then H_1 and H_2 form a Colorful Helly-type Hypergraph Pair.

Quantitative Abstract Helly Type Theorems

Quantitative Colorful Helly Theorems are stable.

Jung, N.

$H_1, H_2 \subseteq 2^X$ – a Colorful Helly-type Hypergraph pair.

Then there is a $\beta : \left(1 - \frac{1}{r_2^{r_1}}, 1\right) \to (0,1)$:

For any $Y \subseteq X$, any partition $Y = \cup_{i=1}^{r_1} Y_i$ and $\alpha \in \left(1 - \frac{1}{r_2^{r_1}}, 1\right)$, if $|H_1|_{Y_1 \times \ldots \times Y_{r_1}}| \geq \alpha |Y_1 \times \ldots \times Y_{r_1}|$, then there is an $i \in [r_1]$ and a subset $Z \subset Y_i$ with $|Z| \geq \beta(\alpha)|Y_i|$ such that $\binom{Z}{r_2} \subseteq H_2$.
Moreover, $\beta(\alpha) \to 1$ as $\alpha \to 1$.

Part III.
Some proofs

Warm up #1

Behrend's result in any dimension

Assume that for any choice $C_1, \dots, C_{d(d+3)/2} \in \mathcal{C}$, the intersection $\bigcap_{i=1}^{d(d+3)/2} C_i$ contains an ellipsoid of volume 1.

Then $\bigcap_{C \in \mathcal{C}} C$ contains an ellipsoid of volume 1.

Equivalently,

Assume that the max. vol. ellipsoid in $\bigcap_{C \in \mathcal{C}} C$ is of volume 1.

Then there is a choice $C_1, \dots, C_{d(d+3)/2} \in \mathcal{C}$, such that the max. vol. ellipsoid in $\bigcap_{i=1}^{d(d+3)/2} C_i$ is of volume 1.

Proof. Follows immediately from John's theorem. □

Order ellipsoids

Definition

$\mathcal{E}$ – an ellipsoid. The *height* of $\mathcal{E}$ is the largest dth coordinate.

C – a convex body containing an ellipsoid of volume 1.
Then there is a unique $\mathcal{E}$ of volume 1 such that every other ellipsoid of volume 1 in C has larger height.
We call this the *lowest ellipsoid* in C.

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Key Lemma

$C_1, \ldots, C_{d(d+3)/2}$ – convex bodies in $\mathbb{R}^d$. $K := \bigcap_{i=1}^{d(d+3)/2} C_i$, and

$K_j := \bigcap_{i=1, i\neq j}^{d(d+3)/2} C_i$. $\mathcal{E}$ – the lowest ellipsoid in K.
Then there exists a j such that $\mathcal{E}$ is also the lowest ellipsoid of K_j.

Follows from general Behrend: Add to the list the top support half-space of $\mathcal{E}$.

Warm up #2

Damásdi, Földvári, N.: Colorful generalization of Behrend's result

$\mathcal{C}_1, \dots, \mathcal{C}_{d(d+3)/2}$ – families.
Assume that for any colorful selection

$C_1 \in \mathcal{C}_1, \dots, C_{d(d+3)/2} \in \mathcal{C}_{d(d+3)/2}$ the intersection $\bigcap_{i=1}^{d(d+3)/2} C_i$ contains an ellipsoid of volume 1.

Then for some j, the intersection $\bigcap_{C \in \mathcal{C}_j} C$ contains an ellipsoid of volume 1.

Warm up #2

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Then for some j, the intersection $\bigcap_{C \in \mathcal{C}_j} C$ contains an ellipsoid of volume 1.

Proof: Following Lovász' colorful argument

- For each colorful selection, there is a lowest ellipsoid of volume 1.
- Take the colorful selection with the highest lowest ellipsoid. $\mathcal{E} \subset \bigcap_{i=1}^{d(d+3)/2} C_i$
- The Key Lemma yields a $j \in [d(d+3)/2]$. Now, $\cap \mathcal{C}_j \supset \mathcal{E}$. $\square$

Main colorful result

Damásdi, Földvári, N.: Few color classes

$\mathcal{C}_1, \dots, \mathcal{C}_{3d}$ – families.
Assume that for any colorful choice, $C_{i_1} \in \mathcal{C}_{i_1}, \dots, C_{i_{2d}} \in \mathcal{C}_{i_{2d}}$, the intersection $\bigcap_{k=1}^{2d} C_{i_k}$ contains an ellipsoid of volume 1.
Then, there exists an $i \in [3d]$ such that $\bigcap_{C \in \mathcal{C}_i} C$ contains an ellipsoid of volume $d^{-O(d^2)}$.

Proof of Main Colorful Result: a Lemma

Lemma

$\mathbf{B}^d \subset C$ max. volume ellipsoid.
$\mathcal{E}$ – another ellipsoid in C, $\operatorname{vol}(\mathcal{E}) \geq \delta$.
Then there is a translate of $\frac{\delta}{d^d}\mathbf{B}^d$ which is contained in $\mathcal{E}$.

Proof.

Length of the d semi-axes of $\mathcal{E}$: $a_1, \ldots, a_d$.
$\mathcal{E} \subset C \subset d\mathbf{B}^d$, so no a_i is too large.
$\operatorname{vol}(\mathcal{E}) \geq \delta$, so $\prod a_i$ not too small.
So, no a_i is too small. □

Note: This lemma is the reason for $d^{O(d^2)}$ in the Theorem.

Proof of Main Colorful Result

- Consider the lowest ellipsoid of volume 1 in all colorful choices of $2d - 1$ sets. May assume that the highest one is $\mathbf{B}^d$.
- $\mathbf{B}^d \subset C_1 \cap \ldots \cap C_{2d-1}$.
- H_1 denotes the support half-space of $\mathbf{B}^d$.
- So, $\mathbf{B}^d$ is the max. volume ellipsoid in $M := C_1 \cap \ldots \cap C_{2d-1} \cap H_1$.
- Next, take an arbitrary colorful choice $C_{2d} \in \mathcal{C}_{2d}, C_{2d+1} \in \mathcal{C}_{2d+1}, \ldots, C_{3d} \in \mathcal{C}_{3d}$ of the remaining $d + 1$ families.
- Easy: The intersection of any $2d$ sets of

$$C_1, \ldots, C_{2d-1}, H_1, C_{2d}, \ldots, C_{3d}$$

  contains an ellipsoid of volume 1.
- By [N16],

$$\bigcap_{i=1}^{3d} C_i \cap H_1$$

  contains an ellipsoid $\mathcal{E}$ of vol $\geq \delta := d^{-O(d)}$. Now, $\mathcal{E} \subset M$.

Proof of Main Colorful Result cont.'d

- By [N16],

$$\bigcap_{i=1}^{3d} C_i \cap H_1$$

contains an ellipsoid $\mathcal{E}$ of vol $\geq \delta := d^{-O(d)}$. Now, $\mathcal{E} \subset M$.

- Since $\mathbf{B}^d \subset M$ is the max. volume ellipsoid, by the Lemma, there is a translate of $\frac{\delta}{d^d}\mathbf{B}^d$ which is contained in $\mathcal{E}$ and thus in $\bigcap_{i=2d}^{3d} C_i$.
- Thus, we have shown: for any colorful choice of the remaining $d+1$ families,

$$\bigcap_{i=2d}^{3d} C_i$$

contains a translate of the same convex body $\frac{\delta}{d^d}\mathbf{B}^d$.

- Finally, use Colorful Helly theorem for $C_i \sim \frac{\delta}{d^d}\mathbf{B}^d$.

Proof of Main Colorful Result cont.'d

- By [N16],

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contains an ellipsoid $\mathcal{E}$ of vol $\geq \delta := d^{-O(d)}$. Now, $\mathcal{E} \subset M$.

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- Finally, use Colorful Helly theorem for $C_i \sim \frac{\delta}{d^d}\mathbf{B}^d$.

Thank you!