Enflo's problem

Paata Ivanishvili

joint work with Ramon van Handel and Sasha Volberg.

NC State University

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Stefan Banach

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Banach algebra; Amenable Banach algebra; Banach Jordan algebra; Banach function algebra; Banach *-algebra; Banach algebra cohomology; Banach bundle; Banach fixed-point theorem; Banach game; Banach lattice; Banach limit; Banach–Mazur compactum; Banach manifold; Banach measure; Banach norm; Banach space; Banach–Alaoglu theorem; Banach–Mazur compactum; Banach–Mazur game; Banach–Ruziewicz problem; Banach–Saks theorem; Banach–Schauder theorem; Banach–Steinhaus theorem; Banach–Stone theorem; Banach–Tarski paradox; Banach's matchbox problem; Hahn–Banach theorem;

Type and Enflo type: linear vs nonlinear.

X has **Rademacher type** $p > 1$ if $\exists C > 0$ such that

$$\mathbb{E}\| \sum_{1\leq j\leq n} \varepsilon_j x_j\|^p \leq C \sum_{1\leq j\leq n} \|x_j\|^p$$

for all $x_1, \dots, x_n \in X$, any $n \geq 1$, $\varepsilon_j = \pm 1$ i.i.d. Rademacher functions.
remark: reverse inequality is called cotype.

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Question (P. Enflo): What about the converse?

Per Enflo

Stanislaw Mazur and Per Enflo

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- Mendel–Naor: Rademacher p, implies "Scaled" Enflo type p.

- Eskenazis: "Relaxed" UMD and Rademacher p, imply Enflo p.

Theorem (P.I., R. van Handel, S. Volberg)

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Proof:

$$\mathbb{E}\|f(\varepsilon)-f(-\varepsilon)\|^p \leq C\,\mathbb{E}\sum_{1\leq j\leq n}\|f(\varepsilon_1,\ldots,\varepsilon_j,\ldots,\varepsilon_n)-f(\varepsilon_1,\ldots,-\varepsilon_j,\ldots,\varepsilon_n)\|^p$$

for **linear** f implies for **all** f.

Not clear how to start...

Let $p = 2$.

Pisier and Poincaré inequality

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where

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remark: $\mathbb{E}\|f(\varepsilon) - f(-\varepsilon)\|^2 \leq 4\mathbb{E}\|f - \mathbb{E}f\|^2$.

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2-uniformly smooth Banach spaces.

Observation 2): For $X = \ell_1^n$ and $f(\varepsilon) = \varepsilon_1 e_1 + ... + \varepsilon_n e_n$

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One can show

$$\mathbb{E}\|f - \mathbb{E}f\|^2 \leq Cn\mathbb{E}\sum \|D_j f(\varepsilon)\|^2$$

for an arbitrary Banach space.

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$$= \frac{\pi}{2}\|\langle \nabla g(X), Y\rangle\|_2 = \frac{\pi}{2}\left\|\sum Y_j \partial_j g(X)\right\|_2 = \frac{\pi}{2}\|\nabla g\|_2.$$

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$$\|f(\varepsilon) - f(\delta)\|_2 \leq C \left\| \sum \delta_j D_j f(\varepsilon) \right\|_2$$

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$\exists X$ *and* $f : \{-1,1\}^n \to X : \|f-\mathbb{E}f\|_2 \geq C\log(n)\,\|\sum \delta_j D_j f(\varepsilon)\|_2$.

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Theorem (P.I., R. van Handel, A. Volberg)

$\|f-\mathbb{E}f\|_2 \leq C\,\|\sum \delta_j D_j f(\varepsilon)\|_2$ *iff* X *has finite cotype.*

How to prove $\|g(X) - \mathbb{E}g\|_2 \leq C\|\sum_{j=1}^{n} Y_j \partial_j g(X)\|_2$?

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$L := -\Delta + x\cdot\nabla$,

$e^{-Lt}g(x) = \mathbb{E}g(e^{-t}x + \sqrt{1-e^{-2t}}Y), \quad Y \sim \mathcal{N}(0, I_{n\times n})$.

$$g(x)-\mathbb{E}g = -\int_0^\infty \frac{d}{dt}e^{-tL}g(x)dt = \int_0^\infty Le^{-tL}g(x)dt.$$

$$Le^{-tL}g(x) = (-\Delta + x\cdot\nabla)\int_{\mathbb{R}^n} g(e^{-t}x+\sqrt{1-e^{-2t}}y)\frac{e^{-|y|^2/2}}{(2\pi)^{n/2}}dy$$

$$\int_{\mathbb{R}^n}\left[-e^{-2t}\Delta g(e^{-t}x+\sqrt{1-e^{-2t}}y) + \langle e^{-t}x, \nabla g(e^{-t}x+\sqrt{1-e^{-2t}}y)\rangle\right]\frac{e^{-|y|^2/2}}{(2\pi)^{n/2}}dy$$

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$$\int_{\mathbb{R}^n}\left[-\frac{e^{-2t}}{1-e^{-2t}}\Delta_y[g(e^{-t}x+\sqrt{1-e^{-2t}}y)]+\langle e^{-t}x,\nabla g(e^{-t}x+\sqrt{1-e^{-2t}}y)\rangle\right]\frac{e^{-|y|^2/2}}{(2\pi)^{n/2}}dy =$$

$$\int_{\mathbb{R}^n}\left[-\left\langle\frac{e^{-2t}y}{1-e^{-2t}},\nabla_y[g(e^{-t}x+\sqrt{1-e^{-2t}}y)]\right\rangle+\langle e^{-t}x,\nabla g(e^{-t}x+\sqrt{1-e^{-2t}}y)\rangle\right]\frac{e^{-|y|^2/2}}{(2\pi)^{n/2}}dy$$

$$\int_{\mathbb{R}^n}\left\langle e^{-t}x-\frac{e^{-2t}y}{\sqrt{1-e^{-2t}}},\nabla g(e^{-t}x+\sqrt{1-e^{-2t}}y)\right\rangle\frac{e^{-|y|^2/2}}{(2\pi)^{n/2}}dy$$

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$$\|g(X)-\mathbb{E}g\|_2 \leq \int_0^\infty \left\|\left\langle e^{-t}X - \frac{e^{-2t}Y}{\sqrt{1-e^{-2t}}}, \nabla g(e^{-t}X+\sqrt{1-e^{-2t}}Y)\right\rangle\right\|_2 dt =$$
$$\|\langle Y, \nabla g(X)\rangle\|_2 \int_0^\infty \frac{1}{\sqrt{e^{2t}-1}}dt = \|\sum Y_j \partial_j g(X)\|_2 \frac{\pi}{2}$$

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$e^{-t}X - \frac{e^{-2t}Y}{\sqrt{1-e^{-2t}}}$ and $e^{-t}X+\sqrt{1-e^{-2t}}Y$ are independent!

$$g(x)-\mathbb{E}g=\int_0^\infty \mathbb{E}\left\langle \sqrt{1-e^{-2t}}x-e^{-t}Y, \nabla g(e^{-t}x+\sqrt{1-e^{-2t}}Y)\right\rangle \frac{dt}{\sqrt{e^{2t}-1}}$$

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Lemma

For any $f : \{-1,1\}^n \to X$ *we have*

$$f(\varepsilon) - \mathbb{E}f = \int_0^\infty \mathbb{E}_\xi \sum \delta_j(t) D_j f(\varepsilon\xi(t)) \frac{dt}{\sqrt{e^{2t}-1}},$$

where $\delta_j(t) = \frac{\xi_j(t)-e^{-t}}{\sqrt{1-e^{-2t}}}$, $\mathbb{P}(\xi_j(t)=1) = \frac{1+e^{-t}}{2}$ *and* $\mathbb{P}(\xi_j(t)=-1) = \frac{1-e^{-t}}{2}$, *and* $\varepsilon\xi(t) = (\varepsilon_1\xi_1(t), \dots, \varepsilon_n\xi_n(t))$

$\mathbb{E}\|f-\mathbb{E}f\|^2 \leq C\,\mathbb{E}\sum \|D_j f(\varepsilon)\|^2$ Gauss vs hypercube

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$$\|f(\varepsilon)-\mathbb{E}f\|_2 \leq \int_0^\infty \left(\mathbb{E}_\varepsilon \left|\mathbb{E}_\xi \sum \delta_j(t) D_j f(\varepsilon\xi(t))\right|^2\right)^{1/2} \frac{dt}{\sqrt{e^{2t}-1}} \leq$$

$$\int_0^\infty \left(\mathbb{E}_{\varepsilon,\xi} \left|\sum \delta_j(t) D_j f(\varepsilon\xi(t))\right|^2\right)^{1/2} \frac{dt}{\sqrt{e^{2t}-1}} \leq$$

$$\int_0^\infty \left(\mathbb{E}_{\varepsilon,\xi,\xi'} \left|\sum (\delta_j(t)-\delta_j'(t)) D_j f(\varepsilon\xi(t))\right|^2\right)^{1/2} \frac{dt}{\sqrt{e^{2t}-1}} =$$

$$\int_0^\infty \left(\mathbb{E}_{\varepsilon,\xi,\xi',\varepsilon'} \left|\sum \varepsilon_j'(\delta_j(t)-\delta_j'(t)) D_j f(\varepsilon\xi(t))\right|^2\right)^{1/2} \frac{dt}{\sqrt{e^{2t}-1}} \overset{\text{Type 2}}{\leq}$$

$$C\int_0^\infty \left(\mathbb{E}_{\varepsilon,\xi,\xi'} \sum (\delta_j(t)-\delta_j'(t))^2 |D_j f(\varepsilon\xi(t))|^2\right)^{1/2} \frac{dt}{\sqrt{e^{2t}-1}} =$$

$$C\int_0^\infty \left(\mathbb{E}_{\xi,\xi'} \sum (\delta_j(t)-\delta_j'(t))^2 \mathbb{E}_\varepsilon |D_j f(\varepsilon)|^2\right)^{1/2} \frac{dt}{\sqrt{e^{2t}-1}} =$$

$$C\int_0^\infty (\mathbb{E}_{\xi,\xi'} (\delta_1(t)-\delta_1'(t))^2)^{1/2} \left(\sum \mathbb{E}_\varepsilon |D_j f(\varepsilon)|^2\right)^{1/2} \frac{dt}{\sqrt{e^{2t}-1}} =$$

$$C\frac{\pi}{\sqrt{2}} \left(\sum \mathbb{E}|D_j f(\varepsilon)|^2\right)^{1/2}.$$

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where $\Delta = \sum_{k=1}^{n} D_k$, and $e^{-t\Delta} f(\varepsilon) = \mathbb{E}_{\varepsilon'} f(\varepsilon') \prod_{j=1}^{n}(1 + e^{-t}\varepsilon_j \varepsilon'_j)$.

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$$\Delta e^{-t\Delta} f(\varepsilon) = \sum_{k=1}^n D_k \mathbb{E}_{\varepsilon'} f(\varepsilon') \prod_{j=1}^n (1 + e^{-t}\varepsilon_j \varepsilon'_j) =$$

$$\mathbb{E}_{\varepsilon'} \sum_{k=1}^n f(\varepsilon') \frac{e^{-t}\varepsilon_k \varepsilon'_k}{1 + e^{-t}\varepsilon_k \varepsilon'_k} \prod_{j=1}^n (1 + e^{-t}\varepsilon_j \varepsilon'_j) \overset{\eta = \varepsilon\varepsilon'}{=}$$

$$\mathbb{E}_\eta \sum_{k=1}^n f(\eta\varepsilon) \frac{e^{-t}\eta_k}{1 + e^{-t}\eta_k} \prod_{j=1}^n (1 + e^{-t}\eta_j) =$$

$$\mathbb{E}_\eta \sum_{k=1}^n f(\eta\varepsilon) \frac{e^{-t}(\eta_k - e^{-t})}{1 - e^{-2t}} \prod_{j=1}^n (1 + e^{-t}\eta_j) =$$

$$\frac{1}{\sqrt{e^{2t} - 1}} \mathbb{E}_\eta \sum_{k=1}^n f(\eta\varepsilon) \frac{(\eta_k - e^{-t})}{\sqrt{1 - e^{-2t}}} \prod_{j=1}^n (1 + e^{-t}\eta_j) = \frac{1}{\sqrt{e^{2t} - 1}} \mathbb{E}_\xi \sum_{k=1}^n f(\xi(t)\varepsilon)\delta_j(t).$$

The end

$$\mathbb{E}\|f - \mathbb{E}f\|^2 \leq C\mathbb{E} \sum_{1 \leq j \leq n} \|D_j f\|^2$$

if and only if X has type 2.

$$\mathbb{E}\|f - \mathbb{E}f\|^2 \leq C\mathbb{E}\| \sum_{1 \leq j \leq n} \delta_j D_j f(\varepsilon)\|^2$$

if and only if X has finite cotype.

Thank you!