The asymmetric case of Hilbert's fourth problem

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October 12, 2021

Metric formulation of the problem

Construct and study all continuous (possibly asymmetric) metrics on open convex subsets of projective n-space so that projective (straight) lines are geodesic.

Leaving the real projective spaces aside and considering an arbitrary open, convex subset $\mathcal{O} \subset \mathbb{R}^n$, the problem consists constructing and studying all continuous functions

$$d : \mathcal{O} \times \mathcal{O} \longrightarrow \mathbb{R}$$

satisfying the properties

- $d(\mathbf{x}, \mathbf{y}) \geq 0$.
- $d(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$.
- $d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) \geq d(\mathbf{x}, \mathbf{z})$.
- $d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) = d(\mathbf{x}, \mathbf{z})$ whenever $\mathbf{y}$ belongs to the line segment $\mathbf{xz}$.

Examples of projective metrics

Why was Hilbert interested in this problem?

- Foundations of geometry.
- Calculus of variations.

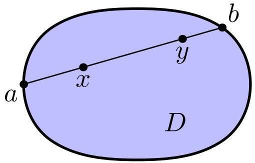
What examples did he have in mind?

- Metrics arising in Minkowski's geometry of numbers.
- Hilbert metrics.

Other examples:

- Funk metrics.

Hilbert and Funk metrics associated to a convex body



Hilbert metric:

$$d(x,y) := \frac{1}{2} \ln \left(\frac{\|y-a\| \|x-b\|}{\|y-b\| \|x-a\|} \right).$$

Funk asymmetric metric:

$$d(x,y) := \ln \left(\frac{\|x-b\|}{\|y-b\|} \right).$$

Busemann's construction of symmetric projective metrics

Let μ be a signed Borel measure on the manifold of all affine hyperplanes intersecting the open, convex set $\mathcal{O} \subset \mathbb{R}^n$, $\mathrm{Graff}_{n-1}(\mathcal{O})$, and assume μ satisfies the following two conditions:

1. The measure of the set of hyperplanes passing through any given point in $\mathcal{O}$ is zero.
2. If $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ ($\mathbf{x} \neq \mathbf{y}, \mathbf{z}$) are any three points in $\mathcal{O}$, the set of hyperplanes separating the point $\mathbf{x}$ from the segment $\mathbf{yz}$ has strictly positive measure.

Given the measure μ define the distance function $d_\mu(\mathbf{x}, \mathbf{y})$ as the measure of the set of all hyperplanes intersecting the segment $\mathbf{xy}$. It is easily verified that the resulting distance is continuous, symmetric, and projective.

Shortcomings of the Busemann construction

1. There is no known modification that can yield (nontrivial) asymmetric metrics.

2. In dimension two, the positivity condition on the measure just means that the measure is positive, but in higher dimensions there does not seem to be a simple characterization of these *quasi-positive measures.*

3. In dimensions greater than two, the construction does not yield all continuous, symmetric projective metrics (Szabó).

Merits of the Busemann construction

Theorem (Pogorelov)

Every continuous, symmetric projective metric in the plane, open convex planar set, or the real projective plane is obtained by Busemann's construction.

Theorem (Pogorelov)

Every smooth, reversible, projective Finsler metric an open convex subset of $\mathbb{RP}^n$ is obtained by Busemann's construction. Moreover, the corresponding measure is smooth and uniquely determined by the metric.

Theorem (Pogorelov, Szabó)

Every continuous, (symmetric or not) projective metric on an open convex subset of $\mathbb{RP}^n$ can be uniformly approximated on compact sets by smooth projective Finsler metrics.

Trivial construction of asymmetric solutions

We are given a set X and a distance function $d : X \times X \to [0, \infty)$. If $f : X \to \mathbb{R}$ is any function satisfying $d(x, y) + f(y) - f(x) > 0$ for all pairs of distinct points $x, y \in X$, then

$$d_f(x, y) := d(x, y) + f(y) - f(x)$$

is again a distance function.

Moreover, three points x, y, and z in X satisfy $d(x, y) + d(y, z) = d(x, z)$ if and only if $d_f(x, y) + d_f(y, z) = d_f(x, z)$. In other words, *the metrics d and d_f have the same geodesics.*

We will consider this way of constructing asymmetric metrics with given geodesics as trivial.

Projective metrics in $\mathbb{RP}^n$

Theorem (Álvarez Paiva)

If d is a continuous asymmetric projective metric on $\mathbb{RP}^n$, there exists a continuous function $f : \mathbb{RP}^n \to \mathbb{R}$ such that $d(x,y) - d(y,x) = f(y) - f(x)$ for every pair of points $x, y \in \mathbb{RP}^n$.

In other words, the *only* way to construct a continuous asymmetric distance function d on $\mathbb{RP}^n$ for which all oriented projective lines are geodesics is to take a symmetric distance function $\tilde{d}$ with the same property, find a continuous function $f : \mathbb{RP}^n \to \mathbb{R}$ such that

$$\tilde{d}(x,y) > f(y) - f(x)$$

for all pairs of distinct points, and set

$$d(x,y) = \tilde{d}(x,y) + f(y) - f(x).$$

Projective Finsler metrics in $\mathbb{RP}^n$

Definition

Let V be a finite-dimensional vector space over the reals. An asymmetric norm $\| \cdot \| : V \to [0, \infty)$ is said to be a *Minkowski norm* if outside the origin the function $\| \cdot \|^2$ is C^2 and its Hessian is positive-definite. A continuous function $F : TM \to [0, \infty)$ defined on the tangent bundle of a smooth manifold M will be said to be a C^k ($k \geq 2$) *Finsler metric* if it is C^k outside the zero section and its restriction to each tangent space is a Minkoswki norm.

Theorem

A smooth projective Finsler metric on the real projective space $\mathbb{RP}^n$ is the sum of a reversible projective Finsler metric and the differential of a smooth function on $\mathbb{RP}^n$.

In two dimensions this result is due to Gautier Berck, in higher dimensions and for all geodesically reversible Zoll Finsler manifolds it is due to Álvarez Paiva.

Holmes-Thompson volume of a Finsler manifold

The restriction of the Finsler metric F to a tangent space $T_x M$ is the support function of the *unit co-disc*

$$D_x^* M := \{\xi_x \in T_x^* M : \xi_x(v_x) \leq 1 \text{ whenever } F(v_x) \leq 1\}.$$

The *unit co-disc bundle* $D^* M \subset T^* M$ of a Finsler metric on a manifold M is the union of all the co-discs $D_x^* M$, $x \in M$, seen as a disc bundle over M.

Definition

The *Holmes-Thompson volume* of a Finsler metric on a manifold M of dimension n is the symplectic volume of its unit co-disc bundle divided by the volume of the n-dimensional Euclidean unit disc. The *area* of a submanifold is defined as the Holmes-Thompson volume of the submanifold provided with its inherited metric.

Holmes-Thompson volume density of a Finsler manifold

Definition

The Holmes-Thompson volume density of a Finsler manifold (M, F) is the function

$$\Phi : \bigwedge^n TM \longrightarrow [0, \infty)$$

that measures the volume of parallelotopes formed by vectors tangent to M: if $v_1, \ldots, v_n$ is a basis of $T_x M$, the n-vector $v_1 \wedge \cdots \wedge v_n$ is a volume form on $T^*_x M$. Denoting the volume of the n-dimensional Euclidean unit disc by ε_n, we define

$$\Phi(v_1 \wedge \cdots \wedge v_n) := \frac{1}{\varepsilon_n} \int_{D^*_x M} |v_1 \wedge \cdots \wedge v_n|.$$

That this is indeed the density for the Holmes-Thompson volume—defined through the symplectic volume of the unit codisc bundle—follows immediately from the computation of the pushfoward of the symplectic volume form on $D^* M$ onto M under the canonical projection $\pi : D^* M \to M$.

Two basic properties of the Holmes-Thompson volume

Proposition

Adding a 1-form to a Finsler metric does not alter its volume density.

Indeed, adding the form merely translates the unit co-discs in each cotangent space.

Definition

The central symmetrization of a Finsler metric F is the Finsler metric defined by $F^S(v_x) = (F(v_x) + F(-v_x))/2$.

Proposition

The Holmes-Thompson volume density of a Finsler metric on a manifold M is less than or equal to that of its central symmetrization. Moreover, the volume densities are equal if and only if $F = F^S + \beta$ for some 1-form β on M.

Central symmetrization of Finsler metrics

Proposition

The Holmes-Thompson volume density of a Finsler metric on a manifold M is less than or equal to that of its central symmetrization. Moreover, the volume densities are equal if and only if $F = F^S + \beta$ for some 1-form β on M.

Proof.

- Note that if $D_x^* M \subset T^* M$ is the convex body supported by the restriction of F to $T_x M$, the restriction of F^S to this vector space is the support function of the symmetrized body $\Delta D_x^* M := (D_x^* M - D_x^* M)/2$.
- Apply the Brunn-Minkowski inequality at every every cotangent space $T_x^* M$, $x \in M$.
- Note that the 1-form will be smooth if the bodies $D_x^* M$ depend smoothly on the base point.

□

Projective metrics in $\mathbb{RP}^n$: strategy of proof

Theorem

If d is a continuous asymmetric projective metric on $\mathbb{RP}^n$, there exists a continuous function $f : \mathbb{RP}^n \to \mathbb{R}$ such that $d(x,y) - d(y,x) = f(y) - f(x)$ for every pair of points $x, y \in \mathbb{RP}^n$.

Theorem (Pogorelov, Szabó)

Every continuous, (symmetric or not) projective metric on an open convex subset of $\mathbb{RP}^n$ can be uniformly approximated on compact sets by smooth projective Finsler metrics.

Theorem

A smooth projective Finsler metric on the real projective space $\mathbb{RP}^n$ is the sum of a reversible projective Finsler metric and the differential of a smooth function on $\mathbb{RP}^n$.

Projective metrics in $\mathbb{RP}^n$: strategy of proof

Theorem

A smooth projective Finsler metric on the real projective space $\mathbb{RP}^n$ is the sum of a reversible projective Finsler metric and the differential of a smooth function on $\mathbb{RP}^n$.

Lemma

If F is a projective Finsler metric on $\mathbb{RP}^n$, then its central symmetrization F^S is a reversible projective Finsler metric on $\mathbb{RP}^n$. Moreover, the length of the projective lines for both metrics is the same.

Lemma

The Holmes-Thompson volume of a projective Finsler metric on $\mathbb{RP}^n$ is an integer multiple of $(\varepsilon_n n!)^{-1}$ times the n-th power of the length of a projective line. Here ε_n is the (Euclidean) volume of the Euclidean unit ball of dimension n.

The upshot

Given a projective Finsler metric F on projective n-space, consider the family of metrics

$$F_\lambda(v_x) := (1-\lambda)F(v_x) + \lambda F(-v_x), \quad 0 \le \lambda \le 1.$$

They are all projective Finsler metrics, the length of projective lines are the same for all values of λ, and $F_{1/2}$ is the central symmetrization of the metric F.

Since the Holmes-Thompson volume of $(\mathbb{RP}^n, F_\lambda)$ is a continuous function of λ and is also an integer multiple of $(\varepsilon_n n!)^{-1}$ times the n-th power of the length of a projective line, it must be constant.

The Holmes-Thompson volume of the metric F is thus equal to that of its central symmetrization F^S, and hence by the equality case of the Brunn-Minkowski inequality $F = F^S + \beta$, where β is a 1-form on $\mathbb{RP}^n$. That β must be closed, and hence exact, is an easy bit of variational calculus. □

Volume of a projective Finsler metric in $\mathbb{RP}^n$

Lemma

The Holmes-Thompson volume of a projective Finsler metric on $\mathbb{RP}^n$ is an integer multiple of $(\varepsilon_n n!)^{-1}$ times the n-th power of the length of a projective line. Here ε_n is the (Euclidean) volume of the Euclidean unit ball of dimension n.

Sketch of the proof. Let $S^*\mathbb{RP}^n$ be the unit cosphere bundle of $(\mathbb{RP}^n, F)$ and let $G_2^+(\mathbb{R}^{n+1})$ denote the Grassmannian of oriented two-dimensional subspaces in $\mathbb{R}^{n+1}$. Note that we can identify $G_2^+(\mathbb{R}^{n+1})$ with the space of oriented geodesics of $(\mathbb{RP}^n, F)$. We have then the circle bundle

$$\pi : S^*\mathbb{RP}^n \longrightarrow G_2^+(\mathbb{R}^{n+1}).$$

We can consider the restriction of the canonical 1-form in T^*M (written as $p \cdot dq$ in local coordinates) to S^*M as a connection in this circle bundle. If T is the (prime) period of the geodesics of $(\mathbb{RP}^n, F)$, then (Boothby and Wang, regular contact manifolds) $(1/T)$ times the curvature $d\alpha$ induces a closed 2-form ω in $G_2^+(\mathbb{R}^{n+1})$ representing an integral cohomology class (i.e., the Euler class of the circle bundle).

Sketch of the proof (cont.)

Note that the fiber integration of the volume form $(1/T)^n \alpha \wedge d\alpha^{n-1}$ on $S^*\mathbb{RP}^n$ yields precisely the form ω^{n-1} in $G_2^+(\mathbb{R}^{n+1})$ and, therefore,

$$\mathrm{vol}(\mathbb{RP}^n, F) = \frac{1}{\varepsilon_n n!} \int_{S^*\mathbb{RP}^n} \alpha \wedge d\alpha^{n-1} = \frac{T^n}{\varepsilon_n n!} \int_{G_2^+(\mathbb{R}^{n+1})} \omega^{n-1} = \frac{T^n}{\varepsilon_n n!} \langle [\omega^{n-1}], [B] \rangle \, .$$

Since $[\omega]$ and, therefore, $[\omega^{n-1}]$ are integral cohomology classes, it follows that $\mathrm{vol}(\mathbb{RP}^n, F)$ is an integer multiple of $(\varepsilon_n n!)^{-1}$ times the n-th power of the length of a projective line. □

Extensions of the theorem

Theorem

If (M, F) is a smooth, geodesically-reversible Zoll Finsler manifold, then F is the sum of a reversible Zoll Finsler metric and the differential of a smooth function on M.

Theorem

The difference of two C^2 projective Finsler metrics on $\mathbb{RP}^n$ $(n > 1)$ that have the same Holmes-Thompson density is the differential of a C^3 function.

Extensions of the theorem

Theorem

Let (M, F) be a reversible Zoll manifold and let

$$L : TM \longrightarrow \mathbb{R}$$

be a continuous function that is homogeneous of degree one and C^2 outside the zero section. If the the geodesics of (M, F) are extremals for the variational problem

$$\gamma \longmapsto \int_\gamma L \ ,$$

then L is the sum of a reversible Lagrangian and the differential of a C^3 function on M.

Other solved "compact" cases of Hilbert's fourth problem

Theorem (Álvarez Paiva and Barbosa Gomes)

Let the distance function d define a possibly asymmetric metric on $\mathbb{R}^n$ for which straight lines are geodesics. If the metric is invariant under a Euclidean crystallographic group and is derived from a C^2 Finsler metric, then there exists a possibly asymmetric norm $\|\cdot\|$ and a function f on $\mathbb{R}^n$ such that

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\| + f(\mathbf{y}) - f(\mathbf{x})$$

for every pair of points $\mathbf{x}$, $\mathbf{y}$ in $\mathbb{R}^n$.

Theorem (Álvarez Paiva)

In two and three dimensions, every C^2 projective Finsler metric on a hyperbolic space form of finite volume is the sum of a multiple of the hyperbolic metric and a closed 1-form.

Hamel's two-dimensional result

Theorem (Hamel, 1903)

Given a positive, continuous function $f : \mathbb{R} \times S^1 \to \mathbb{R}$ *such that*

$$\int_0^{2\pi} e^{-i\theta} f(y\cos(\theta) - x\sin(\theta), \theta)\, d\theta = 0$$

for all real values of x *and* y*, the function defined by*

$$F(x, y; r\cos(\theta), r\sin(\theta)) := r \int_0^{\theta} \sin(\theta - \phi) f(y\cos(\phi) - x\sin(\phi), \phi)\, d\phi$$

is a C^2 *projective Finsler metric on the plane. Moreover any such Finsler metric is of this form modulo the addition of a differential.*

Why isn't Hamel's theorem fully satisfactory?

Hamel's theorem changes the problem to that of finding all positive, continuous function $f : \mathbb{R} \times S^1 \to \mathbb{R}$ such that

$$\int_0^{2\pi} e^{-i\theta} f(y\cos(\theta) - x\sin(\theta), \theta)\, d\theta = 0$$

for all real values of x and y. But that's hard (or at least it seems to be).

A large class of examples of these functions is furnished by those positive, continuous functions $f(p, \theta)$ that also satisfy the symmetry condition

$$f(-p, \theta + \pi) = f(p, \theta)$$

for all values of $(p, \theta) \in \mathbb{R} \times S^1$. However, these are precisely the ones that correspond to reversible Finsler metrics.

Translation-invariant solutions

If $f(p,\theta) = f(\theta)$ the positive, continuous functions satisfying

$$\int_0^{2\pi} e^{-i\theta} f(\theta)\, d\theta = 0$$

(i.e. having no first degree harmonics) are those for which the equation

$$h''(\theta) + h(\theta) = f(\theta)$$

admits periodic solutions.

In this case h can be seen as the support function of an oval. In Hamel's theorem, those functions f correspond to translation invariant solutions of Hilbert's fourth problem (i.e., Minkowski planes).

Radially symmetric solutions

If $f(p,\theta) = f(p)$ we get the radially invariant solutions of Hilbert's fourth problem in the plane. It turns out that the only continuous functions $f(p)$ that satisfy

$$\int_0^{2\pi} e^{-i\theta} f(y\cos(\theta) - x\sin(\theta))\, d\theta = 0$$

for all real values of x and y are even functions (thanks to Fedor Petrov and Mateusz Kwaśnicki for help with this via MathOverflow).

Some more work leads to

Theorem (Àlvarez Paiva)

A C^2 projective Finsler metric in $\mathbb{R}^n$ that is radially symmetric is the sum of a reversible Finsler metric and a differential.