

Preferred directions, curvature, and the bottom of the spectrum

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Abstract

For a first nonconstant eigenfunction of a log-concave diffusion, Klartag's preferred-direction estimate bounds the curvature sampled by its gradient in terms of the mean gradient. We examine the converse and the possibility of a curvature-free lower bound. Under an upper Hessian bound we obtain a reverse estimate, and under a directional Fisher-information bound we obtain a sharper variant. These are averaged, eigenfunction-dependent conclusions; even a preferred-direction quotient arbitrarily close to one does not force a pointwise lower Hessian bound.

Two explicitly solvable one-dimensional families have first eigenfunctions whose preferred-direction quotients tend to zero while their isotropic spectral gaps stay bounded away from zero. One family is even, and tensorization makes the quotient small on the entire first eigenspace in every dimension. A Gaussian regularization preserves this phenomenon and gives smooth uniformly convex potentials with compact resolvent and a common upper Hessian bound. Finally, replacing the first eigenspace by a fixed multiplicative spectral window yields a statement equivalent to the Poincaré formulation of the KLS assertion. We also discuss Gaussian rigidity at the opposite endpoint and a possible, but unproved, passage to uniform measures on convex bodies.

1 Introduction

Let μ be a log-concave probability measure on $\mathbb{R}^n$, and let A be the nonnegative operator associated with the Dirichlet form

$$\mathcal{E}_\mu(f, g) = \int \langle \nabla f, \nabla g \rangle d\mu.$$

For a smooth positive density e^{-V} on $\mathbb{R}^n$, the operator is $A = -\Delta + \langle \nabla V, \nabla \rangle$. Write $\lambda = C_P(\mu)^{-1}$ for its spectral gap. If $Af = \lambda f$, the quantity of interest is

$$Q_\mu(f) = \frac{|\int \nabla f d\mu|^2}{\int |\nabla f|^2 d\mu}. \quad (1.1)$$

It measures the extent to which the gradient points in a common direction. The numerator is a vector average, so this question differs from estimating the size of the gradient itself. Jensen's inequality gives $0 \leq Q_\mu(f) \leq 1$. For an isotropic measure and a first eigenfunction, the stronger elementary bound $Q_\mu(f) \leq \lambda \leq 1$ holds.

1.1 Background and the question of a converse

The connection between convexity and spectral gaps has a long history. The Brascamp–Lieb inequality [3] makes the inverse Hessian of a potential an explicit weight in a variance estimate; in particular, $\nabla^2 V \geq \kappa \text{Id}$ implies $C_P(\mu) \leq \kappa^{-1}$. The Kannan–Lovász–Simonovits problem [5] asks, in

its Poincaré formulation, for a universal bound on C_P for isotropic log-concave measures. Unlike the Brascamp–Lieb estimate, such a bound would not require uniform convexity.

A qualitative preferred-direction theorem already appears in Klartag’s 2009 paper on Berry–Esseen inequalities [6, Corollary 1]. For a smooth uniformly strictly convex body, a first nonconstant Neumann eigenfunction has nonzero integral of its gradient. The same argument bounds the multiplicity of that eigenvalue by the dimension. The remark following the corollary credits Friedlander with removing the uniform strict convexity assumption; the details of that extension are not supplied there. This qualitative result should be distinguished from a uniform positive lower bound for (1.1).

Klartag’s 2023 paper [7, Proposition 2.3] gives a quantitative log-concave version. For a first eigenfunction, under the regularity assumptions used there,

$$\int \langle \nabla^2 V \nabla f, \nabla f \rangle d\mu \leq \lambda \left| \int \nabla f d\mu \right|^2. \quad (1.2)$$

The proof combines the integrated Bochner identity with the Poincaré inequality applied to the derivatives of f . Together with the coordinate test, this yields the improved log-concave Lichnerowicz estimate $C_P(\mu) \leq \sqrt{\|\text{Cov}(\mu)\|_{\text{op}}/\kappa}$ when $\nabla^2 V \geq \kappa \text{Id}$. A curvature-free preferred-direction lower bound is formulated in Livshyts’s lecture notes [8, Conjecture 7.2], in a smooth tempered-eigenfunction setting; see also Questions 7.12–7.15 there. It is this stronger quantitative question, not the qualitative convex-body theorem, that the counterexamples below address.

There are two natural converse questions. Does a large mean gradient force positive curvature? And, in the absence of a curvature assumption, does the first eigenspace nevertheless contain a function with a universally large mean gradient? The answers depend on what one means by curvature and on how narrowly one localizes the spectrum.

1.2 Main results

For a smooth potential, define the curvature seen by f by

$$\mathcal{K}_\mu(f) = \frac{\int \langle \nabla^2 V \nabla f, \nabla f \rangle d\mu}{\int |\nabla f|^2 d\mu}. \quad (1.3)$$

The first result gives a converse in precisely this averaged sense.

Theorem A (An averaged converse). *Suppose that $\mu = e^{-V} dx$ is a smooth log-concave probability measure on $\mathbb{R}^n$, that $0 \leq \nabla^2 V \leq M \text{Id}$, and that the spectral gap $\lambda > 0$ is attained. Every nonzero first eigenfunction satisfies*

$$\frac{\lambda^2}{M} Q_\mu(f) \leq \mathcal{K}_\mu(f) \leq \lambda Q_\mu(f). \quad (1.4)$$

If μ is isotropic and $Q_\mu(f) \geq q$, then $\mathcal{K}_\mu(f) \geq q^3/M$.

The upper estimate is (1.2). The reverse estimate follows by integrating the differentiated eigenvalue equation. In section 3 we also replace M by the Fisher information in the direction of the mean gradient, under explicit integrability assumptions, and identify the exact nonnegative defect in the upper estimate.

The word “averaged” cannot be removed. There are even isotropic measures with $Q_\mu(f) \rightarrow 1$, a common upper Hessian bound, and $\inf_x V''(x) = 0$. Thus even near-maximal alignment does not imply uniform convexity. At the same time, near-maximal alignment does force closeness to a

Gaussian in Wasserstein distance; this follows from the Stein-discrepancy estimates of Courtade, Fathi, and Pananjady [4].

The second result rules out a universal preferred-direction estimate on the first eigenspace.

Theorem B (Small preferred directions in a regular class). *For every $n \geq 1$ and every $\eta > 0$, there is an isotropic unconditional probability measure $\mu = e^{-V} dx$ on $\mathbb{R}^n$ with $V \in C^\infty$, compact resolvent, and*

$$0 < \kappa_\mu \text{Id} \leq \nabla^2 V \leq 4\text{Id}, \quad \lambda(\mu) > \frac{1}{2},$$

such that $Q_\mu(f) < \eta$ for every nonzero first eigenfunction f . The potential and all its derivatives have at most polynomial growth. The constant κ_μ is not uniform in this family.

The underlying model is particularly simple:

$$p_a(x) = Z_a^{-1} \text{sech}^a x, \quad a > 2.$$

Its first eigenfunction is $\sinh x$, its gap is $a - 1$, and

$$Q_{p_a}(\sinh) = \frac{Z_{a-1}^2}{Z_a Z_{a-2}} \sim \frac{\pi^2}{4}(a-2) \quad (a \downarrow 2). \quad (1.5)$$

After isotropic rescaling, the gap tends to $\pi^2/12$. A second model, the logarithm of a Gamma random variable, has the exact quotient $(a-2)/(a-1)$ and an isotropic gap tending to $\pi^2/6 - 1$. Both potentials are strictly convex. Their ground-state transforms are the classical hyperbolic Pöschl–Teller and Morse operators, respectively [9, 11]. We give direct proofs of the first-gap statements, so the counterexamples do not depend on an imported spectral formula.

The regularity assertion in [theorem B](#) requires more than merely approximating densities. In [section 6](#) we add a small Gaussian confinement and prove convergence of the isolated first eigenfunction by a min–max argument. This supplies convergence of its mean gradient. Tensorization then treats the entire first eigenspace, not just a selected eigenfunction.

The failure of a first-eigenspace assertion does not preclude a useful low-frequency statement. Let P be the spectral resolution of A and put

$$\mathcal{H}_L = \text{Ran } P((0, L\lambda]), \quad L \geq 1.$$

This definition makes sense even when the gap is not an eigenvalue.

Theorem C (Spectral packets and the KLS assertion). *The following universal assertions are equivalent:*

- (i) *There is $K < \infty$ such that $C_P(\mu) \leq K$ for every isotropic log-concave probability measure in every dimension.*
- (ii) *There are $L < \infty$ and $q > 0$ such that, for every such measure, some nonzero $f \in \mathcal{H}_L$ satisfies $Q_\mu(f) \geq q$.*

Assertion (ii) implies $C_P(\mu) \leq L/q$. Conversely, (i) gives (ii) with $L = TK$ and $q = (1 - 1/T)/K$ for every $T > 1$.

The proof is a coordinate spectral-measure argument, valid for the natural Neumann realization on a convex support as well as for smooth full-support measures. We give equivalent quasimode and

aggregate formulations and two sum rules in the discrete-spectrum case. These are reformulations, not estimates proving the KLS assertion.

The note concludes with a possible route from the one-dimensional examples to uniform measures on high-dimensional convex bodies. The geometric construction is elementary; the required spectral convergence is not proved here. Keeping that distinction explicit is important: a counterexample for a weighted diffusion does not automatically give one for a Euclidean Neumann Laplacian.

2 Forms, eigenfunctions, and coordinate tests

We use real-valued functions throughout. For a full-dimensional log-concave probability measure, let $H^1(\mu)$ denote the natural weighted Sobolev form domain. Equivalently for the purposes here, take the closure of restrictions of ambient functions in $C_c^\infty(\mathbb{R}^n)$ in the norm

$$\|f\|_{H^1(\mu)}^2 = \int f^2 d\mu + \int |\nabla f|^2 d\mu.$$

The closed form determines a nonnegative self-adjoint operator A . For uniform measure on a convex body this is the Neumann, not the Dirichlet, realization. Constants are the kernel. We write

$$L_0^2(\mu) = \left\{ f \in L^2(\mu) : \int f d\mu = 0 \right\}, \quad \lambda = \inf \sigma(A|_{L_0^2(\mu)}) = C_P(\mu)^{-1}.$$

For the measures under consideration $\lambda > 0$; finiteness of the Poincaré constant for a fixed full-dimensional log-concave measure is classical, see [2]. No discreteness assumption is implicit in this notation. A *first eigenfunction* is a nonzero $f \in \text{Dom } A$ satisfying $Af = \lambda f$; such a function need not exist.

Every affine function belongs to the form domain, by smooth cutoff approximation and the finite second moment. It need not belong to $\text{Dom } A$. This distinction matters at a boundary.

Lemma 2.1 (Coordinate identity). *Suppose that μ is centered and has finite covariance. For $f \in \text{Dom } A$,*

$$\int \partial_i f d\mu = \mathcal{E}_\mu(f, x_i) = \langle Af, x_i \rangle_{L^2(\mu)}. \quad (2.1)$$

If $Af = sf$, then

$$\int \nabla f d\mu = s \int x f d\mu. \quad (2.2)$$

If μ is isotropic, then

$$\left| \int \nabla f d\mu \right|^2 \leq \|Af\|_2^2. \quad (2.3)$$

In particular, an eigenfunction at eigenvalue $s > 0$ satisfies $Q_\mu(f) \leq s$, and a first eigenfunction satisfies

$$Q_\mu(f) \leq \lambda \leq 1. \quad (2.4)$$

Proof. Equation (2.1) is the defining operator-form identity, tested against the form-domain function x_i . Isotropy says that $x_1, \dots, x_n$ are orthonormal in $L^2(\mu)$, so Bessel's inequality gives (2.3). For an eigenfunction, $\mathcal{E}(f, f) = s\|f\|_2^2$, which gives $Q(f) \leq s$. Testing the Poincaré inequality on x_i gives $\lambda \leq 1$. $\square$

Here and below $Q(f)$ is defined for any form-domain function of positive energy by (1.1); adding a constant or multiplying by a nonzero scalar does not change it.

Lemma 2.2 (One-dimensional standardization). *Let X have mean m and variance $s^2 > 0$, and let ν be the law of $Y = (X - m)/s$. For $g(y) = f(m + sy)$,*

$$Q_\nu(g) = Q_\mu(f), \quad \lambda(\nu) = s^2 \lambda(\mu).$$

If $d\mu = e^{-V} dx$, the standardized potential has second derivative $s^2 V''(m + sy)$.

Proof. The pullback is unitary on L^2 , while $g' = sf'(m + s \cdot)$. The mean derivative is multiplied by s , the energy by s^2 , and the Rayleigh quotient by s^2 . $\square$

This invariance is only being asserted for one-dimensional affine standardization (and, more generally, similarities), not for arbitrary anisotropic linear changes of variables.

3 A reverse estimate for the curvature seen by an eigenfunction

We first work with a smooth convex potential V on $\mathbb{R}^n$, with $\mu = e^{-V} dx$ a probability measure. Let f be a first eigenfunction and write

$$h = \nabla f, \quad b = \int h \, d\mu, \quad E_f = \int |h|^2 \, d\mu, \quad H = \nabla^2 V.$$

The central distinction is between a lower bound on $H(x)$ for every x and a lower bound on its quadratic form averaged against h .

Theorem 3.1 (Two-sided curvature estimate). *Assume $0 \leq H \leq M \text{Id}$ with $M < \infty$ and $\lambda > 0$. If $Af = \lambda f$ and $f \neq 0$, then*

$$\frac{\lambda^2}{M} Q_\mu(f) \leq \mathcal{K}_\mu(f) \leq \lambda Q_\mu(f). \quad (3.1)$$

Moreover,

$$\lambda Q_\mu(f) = \mathcal{K}_\mu(f) + \mathcal{D}_\mu(f), \quad \mathcal{D}_\mu(f) = \frac{\int \|\nabla^2 f\|_{\text{HS}}^2 \, d\mu - \lambda \int |\nabla f - b|^2 \, d\mu}{E_f} \geq 0. \quad (3.2)$$

Proof. The integrated Bochner identity gives

$$\int \|\nabla^2 f\|_{\text{HS}}^2 \, d\mu + \int \langle Hh, h \rangle \, d\mu = \int (Af)^2 \, d\mu = \lambda E_f. \quad (3.3)$$

Apply the Poincaré inequality to each component of h and add:

$$\int \|\nabla^2 f\|_{\text{HS}}^2 \, d\mu \geq \lambda(E_f - |b|^2). \quad (3.4)$$

Substitution in (3.3) proves the upper bound and the defect identity. This is Klartag's argument.

For the reverse bound, differentiation of $Af = \lambda f$ gives

$$Ah + Hh = \lambda h. \quad (3.5)$$

Integrating componentwise and testing against the constant function yields

$$\int Hh \, d\mu = \lambda b. \quad (3.6)$$

Since $0 \leq H \leq M\text{Id}$, Jensen's inequality and $H^2 \leq MH$ imply

$$\lambda^2 |b|^2 = \left| \int Hh \, d\mu \right|^2 \leq \int |Hh|^2 \, d\mu \leq M \int \langle Hh, h \rangle \, d\mu.$$

Division by ME_f proves the lower bound.

For completeness, the identities extend to operator-domain eigenfunctions under the stated bounded-Hessian assumption. The drift ∇V is globally Lipschitz and has at most linear growth. Smooth compactly supported functions form an operator core; this also follows by the ground-state transform, whose smooth Schrödinger potential is bounded below, followed by the standard cutoff argument. For a graph-norm approximating sequence, the Bochner identity applied to differences shows that the Hessians are Cauchy in $L^2(\mu)$, since $H \geq 0$. Thus $h \in H^1(\mu; \mathbb{R}^n)$ and (3.3) passes to the limit. Equation (3.5) holds first in distributions. Its right-hand side is in L^2 because H is bounded, so each $h_i \in \text{Dom } A$. Testing with $1 \in H^1(\mu)$ proves (3.6). These are the usual diffusion form identities; see also [1]. $\square$

Corollary 3.2. *Under the hypotheses of theorem 3.1, suppose in addition that μ is isotropic. If $Q_\mu(f) \geq q > 0$, then*

$$\lambda \geq q, \quad C_P(\mu) \leq q^{-1}, \quad \mathcal{K}_\mu(f) \geq q^3/M.$$

If instead $\nabla^2 V \geq \kappa \text{Id}$ is known, then

$$Q_\mu(f) \geq \kappa/\lambda, \quad \lambda^2 \geq \kappa.$$

Proof. Use (2.4) and (3.1). For the last assertions, $\mathcal{K}_\mu(f) \geq \kappa$ and $Q_\mu(f) \leq \lambda$. $\square$

3.1 A directional Fisher-information hypothesis

A pointwise upper Hessian bound is convenient but stronger than the proof needs. The next statement isolates a weaker sufficient condition. Its analytic hypotheses are made explicit so that no unproved integration by parts is hidden in the formulation.

Proposition 3.3 (Directional and matrix variants). *Suppose that V is smooth and convex, $Af = \lambda f$, and the Bochner and integrated commutation identities (3.3) and (3.6) hold with finite integrals. Suppose also that $\partial_i f \in H^1(\mu)$ for every i . If $b \neq 0$, put $\theta = b/|b|$ and*

$$J_\theta = \int \langle H\theta, \theta \rangle \, d\mu < \infty.$$

Then $J_\theta > 0$ and

$$\frac{\lambda^2}{J_\theta} Q_\mu(f) \leq \mathcal{K}_\mu(f) \leq \lambda Q_\mu(f). \quad (3.7)$$

If the matrix $\overline{H} = \int H \, d\mu$ is finite and positive definite, then, more precisely,

$$\int \langle Hh, h \rangle \, d\mu \geq \lambda^2 \langle \overline{H}^{-1}b, b \rangle. \quad (3.8)$$

Whenever integration by parts is justified, $J_\theta = \int (\partial_\theta V)^2 \, d\mu$ is the directional Fisher information.

Proof. Apply Cauchy–Schwarz in $L^2(\mu; \mathbb{R}^n)$ to $H^{1/2}\theta$ and $H^{1/2}h$. By (3.6),

$$\lambda^2|b|^2 = \left(\int \langle H\theta, h \rangle d\mu \right)^2 \leq J_\theta \int \langle Hh, h \rangle d\mu.$$

This also shows $J_\theta > 0$. The upper bound is unchanged. For any constant vector v , nonnegativity of H gives

$$0 \leq \int \langle H(h-v), h-v \rangle d\mu = \int \langle Hh, h \rangle d\mu - 2\lambda \langle b, v \rangle + \langle \bar{H}v, v \rangle.$$

Choose $v = \lambda \bar{H}^{-1}b$. Finally, integrating $\partial_\theta((\partial_\theta V)e^{-V})$ gives the Fisher identity when the boundary term vanishes. $\square$

Remark 3.4. The conclusion does not say that V is uniformly convex, nor even that its Hessian is large in the fixed direction θ at every point. The quantity being controlled is the Hessian sampled by the varying vector $\nabla f(x)$, with the energy of that eigenfunction as normalization.

4 An even explicitly solvable family

For $a > 0$, consider

$$d\mu_a(x) = p_a(x) dx, \quad p_a(x) = Z_a^{-1}(\cosh x)^{-a}, \quad Z_a = \sqrt{\pi} \frac{\Gamma(a/2)}{\Gamma((a+1)/2)}. \quad (4.1)$$

The normalization follows from the beta integral. The potential is even, smooth, and strictly convex:

$$V'_a(x) = a \tanh x, \quad V''_a(x) = a \operatorname{sech}^2 x > 0.$$

It is not uniformly convex. Its linear tails will be important.

Theorem 4.1 (The first mode of the hyperbolic model). *If $a > 2$, the spectral gap of μ_a is $\lambda_a = a - 1$, it is a simple isolated eigenvalue, and its eigenspace is spanned by $f_a(x) = \sinh x$. Furthermore,*

$$\int f_a^2 d\mu_a = \frac{1}{a-2}, \quad \int (f'_a)^2 d\mu_a = \frac{a-1}{a-2}, \quad (4.2)$$

$$Q_{\mu_a}(f_a) = \frac{Z_{a-1}^2}{Z_a Z_{a-2}} = \frac{(a-1)(a-2)}{4} \left(\frac{\Gamma((a-1)/2)}{\Gamma(a/2)} \right)^4. \quad (4.3)$$

In particular,

$$Q_{\mu_a}(f_a) \sim \frac{\pi^2}{4}(a-2) \longrightarrow 0 \quad (a \downarrow 2). \quad (4.4)$$

Proof. Direct differentiation gives

$$A_a f_a = -f''_a + a \tanh x f'_a = (a-1)f_a.$$

The condition $a > 2$ is exactly the condition that both $\sinh x$ and $\cosh x$ belong to $L^2(\mu_a)$. Cutoff approximation puts f_a in the form domain, and the displayed differential equation, tested against compactly supported smooth functions and then against the form domain, shows that $f_a \in \operatorname{Dom} A_a$.

To prove that this is the first eigenvalue, apply the unitary ground-state transform $U_a f = p_a^{1/2} f$. It sends A_a to

$$H_a = -\frac{d^2}{dx^2} + \frac{a^2}{4} - \frac{a(a+2)}{4} \operatorname{sech}^2 x = D_a^* D_a, \quad D_a = \frac{d}{dx} + \frac{a}{2} \tanh x. \quad (4.5)$$

The partner factorization is

$$D_a D_a^* = H_{a-2} + a - 1. \quad (4.6)$$

For $a > 2$, $H_{a-2} = D_{a-2}^* D_{a-2} \geq 0$. The positive spectra of $D_a^* D_a$ and $D_a D_a^*$ agree, by polar decomposition. Thus A_a has no spectrum in $(0, a-1)$. The exhibited eigenfunction attains the lower bound. The kernel of H_{a-2} is one-dimensional, spanned by $\cosh^{-(a-2)/2} x$, so the eigenvalue $a-1$ is simple. The essential spectrum of H_a starts at $a^2/4$, because its potential is a decaying bounded perturbation of $a^2/4$. Since

$$\frac{a^2}{4} - (a-1) = \frac{(a-2)^2}{4} > 0, \quad (4.7)$$

the eigenvalue is isolated. The elementary spectral facts used here may also be found in [10].

Next,

$$\mathbb{E}_{\mu_a} \cosh^2 X = \frac{Z_{a-2}}{Z_a} = \frac{a-1}{a-2}, \quad \mathbb{E}_{\mu_a} \sinh^2 X = \frac{1}{a-2}.$$

Also $\mathbb{E}_{\mu_a} \cosh X = Z_{a-1}/Z_a$. These identities give (4.2) and the first expression in (4.3); the Gamma recurrence gives the second. Finally, $Z_a \rightarrow Z_2 = 2$, $Z_{a-1} \rightarrow Z_1 = \pi$, and $Z_{a-2} \sim 2/(a-2)$, which proves (4.4). $\square$

For this eigenfunction the curvature numerator is especially transparent:

$$\int V_a''(f_a)^2 d\mu_a = a, \quad \mathcal{K}_{\mu_a}(f_a) = \frac{a(a-2)}{a-1} \rightarrow 0.$$

Strict convexity at every finite point therefore coexists with vanishing curvature as sampled by the normalized first-mode energy.

4.1 Isotropic normalization and tensorization

Let $s_a^2 = \operatorname{Var}_{\mu_a} X$ and let ν_a be the law of X/s_a . The tilted beta integral gives

$$\mathbb{E}_{\mu_a} e^{tX} = \frac{\Gamma((a+t)/2)\Gamma((a-t)/2)}{\Gamma(a/2)^2}, \quad |t| < a. \quad (4.8)$$

Writing $\psi_1 = (\log \Gamma)''$ for the trigamma function, we obtain

$$s_a^2 = \frac{1}{2} \psi_1(a/2), \quad \lambda(\nu_a) = \frac{a-1}{2} \psi_1(a/2) \rightarrow \frac{\pi^2}{12} \quad (a \downarrow 2). \quad (4.9)$$

The series $\psi_1(u) = \sum_{k \geq 0} (u+k)^{-2}$ is strictly larger than $\int_0^\infty (u+t)^{-2} dt = 1/u$. Hence

$$\lambda(\nu_a) > \frac{a-1}{a} > \frac{1}{2} \quad (a > 2). \quad (4.10)$$

By [theorem 2.2](#), the quotient is unchanged by standardization.

The $L^2(\mu_a)$ -normalized eigenfunction is $\phi_a(x) = \sqrt{a-2} \sinh x$. It and all its derivatives are tempered in the usual weighted sense: for every derivative order r ,

$$|\phi_a^{(r)}(x)| \sqrt{p_a(x)} \leq C_{a,r} e^{-(a/2-1)|x|}.$$

Thus the failure is not caused by an inadmissible eigenfunction when $a > 2$. At $a = 2$, in contrast, $\sinh x$ is no longer in $L^2(\mu_2)$; one must not treat that endpoint as an eigenfunction with quotient zero. On compact sets the normalized modes tend to zero as $a \downarrow 2$, while their unit L^2 mass escapes to the tails. Equation (4.7) records the simultaneous approach to the edge of the essential spectrum.

Proposition 4.2 (The whole first eigenspace). *Let ν be a centered one-dimensional probability measure whose positive spectral gap λ is a simple isolated eigenvalue with normalized eigenfunction g . The first eigenspace of $\nu^{\otimes n}$ is*

$$\text{span}\{g(x_1), \dots, g(x_n)\}.$$

Every nonzero F in this space has $Q_{\nu^{\otimes n}}(F) = Q_\nu(g)$. In particular this applies to each standardized measure ν_a above.

Proof. The product generator is the sum of the coordinate generators. Its first positive spectral value is λ ; a spectral contribution at that value can have exactly one nonconstant factor, and that factor must lie in the one-dimensional first eigenspace. Equivalently, this follows from the equality case in the tensorized Poincaré inequality. If $F = \sum_i c_i g(x_i)$ and $\beta = \int g' d\nu$, then

$$\int |\nabla F|^2 d\nu^{\otimes n} = \lambda \sum_i c_i^2, \quad \left| \int \nabla F d\nu^{\otimes n} \right|^2 = \beta^2 \sum_i c_i^2.$$

Their ratio is $\beta^2/\lambda = Q_\nu(g)$. □

4.2 Large alignment without a pointwise curvature bound

The same family gives a different obstruction at its other parameter endpoint.

Proposition 4.3. *There are even, smooth, isotropic, strictly log-concave probability measures ν_a on $\mathbb{R}$ such that, for their first eigenfunctions,*

$$Q_{\nu_a}(g_a) \longrightarrow 1, \quad 0 < \tilde{V}_a'' \leq \frac{4}{3}, \quad \inf_{y \in \mathbb{R}} \tilde{V}_a''(y) = 0.$$

Consequently no positive pointwise lower Hessian bound can follow from $Q(f) \geq q$, even for fixed $q < 1$ arbitrarily close to one and with a common upper Hessian bound.

Proof. Take the preceding family with $a \rightarrow \infty$, $a \geq 4$. Since $\mathbb{E} \cosh X \geq 1$,

$$1 \geq Q_{\mu_a}(f_a) = \frac{(\mathbb{E} \cosh X)^2}{\mathbb{E} \cosh^2 X} \geq \frac{a-2}{a-1} \longrightarrow 1.$$

The sharp Poincaré inequality tested on x gives $s_a^2 \leq 1/(a-1)$. Thus the standardized potential satisfies

$$\tilde{V}_a''(y) = a s_a^2 \text{sech}^2(s_a y) \leq \frac{a}{a-1} \leq \frac{4}{3},$$

and its infimum is zero. □

4.3 What remains true qualitatively in one dimension

Proposition 4.4. *Let $p = e^{-V}$ be a smooth positive log-concave probability density on $\mathbb{R}$. If its first positive spectral value is an eigenvalue, every real first eigenfunction can be oriented to be strictly increasing. In particular, its mean derivative is nonzero.*

Proof. The one-dimensional oscillation theorem for the natural self-adjoint realization implies that the first eigenfunction has exactly one zero [10]. Choose its sign so that it is negative to the left of that zero and positive to the right. The equation is

$$(pf')' = -\lambda pf.$$

Since $pf \in L^1(dx)$, pf' has finite limits at both ends. Both limits are zero: a nonzero limit would force $\int p(f')^2 dx = \infty$, because $\int_{-\infty}^{\infty} p^{-1} dx$ and $\int_{-\infty}^{\infty} p^{-1} dx$ diverge. Therefore

$$p(x)f'(x) = -\lambda \int_{-\infty}^x f(t)p(t) dt. \quad (4.11)$$

The integral on the right is strictly negative for every finite x : before the zero this is immediate, and after the zero it equals $-\int_x^{\infty} fp$ because $\int f d\mu = 0$. Thus $f'(x) > 0$. Its integral is finite by Cauchy-Schwarz and positive by strict positivity. $\square$

Thus the examples destroy a uniform quantitative statement, not the one-dimensional qualitative statement. They also illustrate why using a higher even eigenfunction with zero mean derivative would not answer the first-eigenfunction question.

5 A logarithmic Gamma model

The even model is useful for symmetry and regularization. A second family gives a particularly simple rational formula for Q . For $a > 0$, let

$$d\eta_a(x) = \frac{1}{\Gamma(a)} e^{ax-e^x} dx. \quad (5.1)$$

Equivalently, e^X has the Gamma distribution with shape a and rate one. The potential $W_a(x) = e^x - ax + \log \Gamma(a)$ is smooth and strictly convex, with $W_a''(x) = e^x$.

Theorem 5.1. *For $a > 2$, the spectral gap of η_a is $a - 1$ and is a simple isolated eigenvalue. A first eigenfunction is*

$$f_a(x) = e^{-x} - \frac{1}{a-1}.$$

It satisfies

$$\int f_a^2 d\eta_a = \frac{1}{(a-1)^2(a-2)}, \quad \int (f_a')^2 d\eta_a = \frac{1}{(a-1)(a-2)}, \quad (5.2)$$

$$Q_{\eta_a}(f_a) = \frac{a-2}{a-1}. \quad (5.3)$$

After centering and variance-one rescaling, the gap is

$$(a-1)\psi_1(a) > \frac{a-1}{a} > \frac{1}{2}, \quad (a-1)\psi_1(a) \longrightarrow \frac{\pi^2}{6} - 1 \quad (a \downarrow 2). \quad (5.4)$$

Proof. The Gamma integral gives

$$\mathbb{E}_{\eta_a} e^{-rX} = \frac{\Gamma(a-r)}{\Gamma(a)}, \quad r < a. \quad (5.5)$$

In particular, the displayed f_a is centered and belongs to the form domain when $a > 2$. Directly,

$$A_a f_a = -f_a'' + (e^x - a)f_a' = (a-1)e^{-x} - 1 = (a-1)f_a.$$

As in the hyperbolic example, cutoff approximation and the weak equation show membership in $\text{Dom } A_a$.

To see that $a-1$ is the gap, change variables $t = e^x$ and write $F(t) = u(\log t)$. The Gamma potential on $(0, \infty)$ is $U(t) = t - (a-1)\log t$ up to an additive constant, so $U''(t) = (a-1)/t^2$. Brascamp–Lieb, initially for smooth test functions and then by form approximation, yields

$$\text{Var}_{\eta_a} u \leq \frac{1}{a-1} \int_0^\infty t^2 (F'(t))^2 \frac{t^{a-1} e^{-t}}{\Gamma(a)} dt = \frac{1}{a-1} \int (u')^2 d\eta_a.$$

The eigenfunction attains equality.

To prove simplicity, write $p = d\eta_a/dx$ and use the weighted Wronskian. If u, v are two eigenfunctions at the same eigenvalue, then $p(uv' - u'v)$ is constant. Its absolute value is bounded by a constant multiple of $p(u^2 + v^2 + (u')^2 + (v')^2)$, an integrable function. Along a sequence tending to infinity this bound tends to zero, so the constant vanishes. Uniqueness for the second-order differential equation implies linear dependence. Isolation also follows from the ground-state Schrödinger potential

$$-\frac{d^2}{dx^2} + \frac{e^{2x}}{4} - \frac{a+1}{2}e^x + \frac{a^2}{4}. \quad (5.6)$$

Its essential spectrum starts at $a^2/4$, while $a-1 < a^2/4$ for $a > 2$; this is the classical Morse operator, see [11].

Formula (5.5) with $r = 1, 2$ gives (5.2) and (5.3). Finally,

$$\mathbb{E} e^{tX} = \frac{\Gamma(a+t)}{\Gamma(a)}, \quad t > -a,$$

so $\mathbb{E}X = \psi(a)$ and $\text{Var } X = \psi_1(a)$. Apply theorem 2.2, the trigamma series bound, and $\psi_1(2) = \pi^2/6 - 1$. $\square$

The normalized eigenfunction is $(a-1)\sqrt{a-2}[e^{-x} - (a-1)^{-1}]$. For $a > 2$ it and all its derivatives, multiplied by $\sqrt{d\eta_a/dx}$, decay exponentially at $-\infty$ and faster than exponentially at $+\infty$. There are three distinct parameter thresholds: the measure exists for $a > 0$, the mean of e^{-X} exists for $a > 1$, and this particular eigenfunction belongs to the form domain only for $a > 2$. Confusing these thresholds would produce a spurious endpoint counterexample.

There is also an exact equality case in Klartag’s averaged estimate. For this model,

$$\int W_a''(f_a')^2 d\eta_a = \frac{1}{a-1}, \quad \mathcal{K}_{\eta_a}(f_a) = a-2 = \lambda_a Q_{\eta_a}(f_a).$$

Indeed, $f_a' - \int f_a' d\eta_a = -f_a$, so the derivative Poincaré inequality is an equality and the defect in (3.2) vanishes. All the identities involved are justified directly by the exponential decay stated above. The directional Fisher information is $\int W_a'' d\eta_a = a$, so theorem 3.3 applies even though W_a'' is unbounded. In particular, equality in the upper curvature estimate alone is not a Gaussian rigidity statement.

6 Gaussian confinement and regular counterexamples

The examples so far have essential spectrum. We now show that neither essential spectrum nor the failure of uniform convexity for an individual measure is necessary for small preferred directions.

Fix $a > 2$ and set, for $\varepsilon > 0$,

$$p_{a,\varepsilon}(x) = Z_{a,\varepsilon}^{-1} \exp\left(-a \log \cosh x - \frac{\varepsilon}{2} x^2\right). \quad (6.1)$$

Write $\lambda_{a,\varepsilon}$ for its gap and choose a normalized first eigenfunction $f_{a,\varepsilon}$.

Lemma 6.1 (Convergence of the isolated first mode). *For each fixed $a > 2$, the operator associated with (6.1) has compact resolvent and a simple first eigenvalue. As $\varepsilon \downarrow 0$,*

$$\lambda_{a,\varepsilon} \longrightarrow a - 1, \quad (6.2)$$

$$Q_{p_{a,\varepsilon}}(f_{a,\varepsilon}) \longrightarrow Q_{\mu_a}(\sinh). \quad (6.3)$$

The variances also converge to s_a^2 .

Proof. Keep a fixed and suppress it from subscripts. On the common Hilbert space $L^2(dx)$, the ground-state transformed operators satisfy

$$H_\varepsilon = H_0 + \frac{a\varepsilon}{2} x \tanh x + \frac{\varepsilon^2}{4} x^2 - \frac{\varepsilon}{2}. \quad (6.4)$$

The potential tends to $+\infty$ at both ends for $\varepsilon > 0$, so the resolvent is compact. One-dimensional eigenvalues are simple. Since $x \tanh x \geq 0$, the quadratic forms obey

$$H_\varepsilon \geq H_0 - \frac{\varepsilon}{2}. \quad (6.5)$$

The min-max principle at the second variational level gives

$$\lambda_\varepsilon \geq (a - 1) - \frac{\varepsilon}{2}. \quad (6.6)$$

Let $u_0 = \sqrt{p_a}$ be the normalized ground state of H_0 , and let $u_1 = \sqrt{p_a} \sqrt{a - 2} \sinh x$ be its normalized first excited state. Both decay exponentially and have finite moments of all orders. The two-dimensional space $\text{span}\{u_0, u_1\}$ therefore belongs to the form domain of every H_ε , and the matrices of the restricted quadratic forms converge to $\text{diag}(0, a - 1)$. Min-max gives $\limsup \lambda_\varepsilon \leq a - 1$. Together with (6.6) this proves (6.2).

The normalized ground states $u_{0,\varepsilon} = \sqrt{p_{a,\varepsilon}}$ converge to u_0 in $L^2(dx)$ by dominated convergence. Let $u_{1,\varepsilon} = \sqrt{p_{a,\varepsilon}} f_{a,\varepsilon}$. They have unit norm and are orthogonal to $u_{0,\varepsilon}$, so $\langle u_{1,\varepsilon}, u_0 \rangle \rightarrow 0$. Write $\mathfrak{h}_0$ for the closed quadratic form of H_0 . Because $a - 1$ is simple and isolated, there is $\Lambda > a - 1$ such that $H_0 \geq \Lambda$ on $\{u_0, u_1\}^\perp$. Write

$$u_{1,\varepsilon} = c_{0,\varepsilon} u_0 + c_{1,\varepsilon} u_1 + r_\varepsilon, \quad r_\varepsilon \perp u_0, u_1.$$

By (6.5),

$$(a - 1)|c_{1,\varepsilon}|^2 + \Lambda \|r_\varepsilon\|_2^2 \leq \mathfrak{h}_0[u_{1,\varepsilon}] \leq \lambda_\varepsilon + \varepsilon/2.$$

Since $c_{0,\varepsilon} \rightarrow 0$ and the total squared norm is one, $\|r_\varepsilon\|_2 \rightarrow 0$ and $|c_{1,\varepsilon}| \rightarrow 1$. Choose the sign so that $c_{1,\varepsilon} \geq 0$. Then

$$u_{1,\varepsilon} \longrightarrow u_1 \quad \text{in } L^2(dx). \quad (6.7)$$

To recover the mean derivative, no convergence of derivatives is needed. The coordinate identity gives

$$\int f'_{a,\varepsilon} p_{a,\varepsilon} dx = \lambda_\varepsilon \langle u_{1,\varepsilon}, x \sqrt{p_{a,\varepsilon}} \rangle_{L^2(dx)}. \quad (6.8)$$

Dominated convergence gives $x \sqrt{p_{a,\varepsilon}} \rightarrow x \sqrt{p_a}$ in $L^2(dx)$, as well as convergence of variances. Equations (6.7) and (6.8) show convergence of the mean derivatives. The energies are λ_ε because the eigenfunctions have unit norm, so (6.3) follows. $\square$

Remark 6.2. The proof concerns a fixed isolated eigenvalue and its spectral vector. It does not claim norm-resolvent convergence of the entire operator. Indeed, compact resolvents cannot converge in operator norm to a noncompact resolvent. Density convergence by itself would also have been insufficient to justify (6.3).

Theorem 6.3. *The measures in theorem B exist. They may be chosen to be products of a single even one-dimensional measure.*

Proof. Choose $2 < a < 3$ sufficiently close to 2 that $Q_{\mu_a}(\sinh) < \eta$. By theorem 6.1, for all sufficiently small $\varepsilon > 0$ the regularized first-mode quotient is still less than η and the isotropically rescaled gap is greater than $1/2$. Also require $\varepsilon < a - 2$.

Let $s_{a,\varepsilon}^2$ be the variance before rescaling. Equation (6.6) implies $\lambda_{a,\varepsilon} > 1$, and the Poincaré inequality tested on x implies $s_{a,\varepsilon}^2 \leq \lambda_{a,\varepsilon}^{-1} < 1$. The standardized potential satisfies

$$\varepsilon s_{a,\varepsilon}^2 \leq \tilde{V}_{a,\varepsilon}''(y) = s_{a,\varepsilon}^2 (a \operatorname{sech}^2(s_{a,\varepsilon} y) + \varepsilon) \leq (a + \varepsilon) s_{a,\varepsilon}^2 < 4.$$

It is smooth; it grows quadratically, its first derivative grows at most linearly, and its higher derivatives are bounded. Thus it belongs to the regular class of [7], with constants allowed to depend on the measure. Its eigenfunctions are tempered by [7, Lemma 2.2].

Take the n -fold product. The measure is isotropic and unconditional, the Hessian bounds hold coordinatewise, and the resolvent is compact. The product gap equals the one-dimensional gap. Finally theorem 4.2 gives the same quotient for every nonzero vector in the first eigenspace. $\square$

There is no contradiction with the preferred-direction estimate under a fixed uniform convexity assumption: by theorem 3.2, a common positive lower Hessian bound would prevent these quotients from tending to zero. The new point is that smoothness, attainment, a common upper Hessian bound, symmetry, and compact resolvent do not replace that quantitative lower bound.

7 The Gaussian endpoint

Small Q is compatible with a good spectral gap. At the other endpoint, Q close to one has a much stronger consequence.

Proposition 7.1 (Rigidity and a stability consequence). *Let $\mu = e^{-V} dx$ be a smooth isotropic probability measure on $\mathbb{R}^n$ with spectral gap $\lambda > 0$ and a first eigenfunction f . If $Q_\mu(f) \geq 1 - \delta$ with $0 \leq \delta < 1$, then*

$$W_2(\mu, \gamma_n)^2 \leq \frac{n\delta}{1 - \delta}, \quad (7.1)$$

where γ_n is standard Gaussian measure. In particular, if $Q_\mu(f) = 1$ then $\mu = \gamma_n$.

Proof. The coordinate bound gives $\lambda \geq Q_\mu(f) \geq 1 - \delta$, hence $C_P(\mu) \leq (1 - \delta)^{-1}$. The spectral-gap/Stein-kernel estimate of Courtade, Fathi, and Pananjady [4] states, for centered isotropic measures with a Poincaré inequality, that

$$W_2(\mu, \gamma_n)^2 \leq n(C_P(\mu) - 1). \quad (7.2)$$

This gives (7.1).

One can see the relevant constant directly from the form construction of a Stein kernel. Set $u_i = A^{-1}x_i$ in $L_0^2(\mu)$ and $\tau_{ij} = \partial_j u_i$. For smooth test functions φ ,

$$\int x_i \varphi d\mu = \int \langle \nabla u_i, \nabla \varphi \rangle d\mu, \quad \int \tau_{ij} d\mu = \delta_{ij}.$$

Moreover,

$$\int \|\tau - \text{Id}\|_{\text{HS}}^2 d\mu = \sum_i \langle x_i, A^{-1}x_i \rangle - n \leq n(\lambda^{-1} - 1).$$

The standard bound of W_2 by the L^2 Stein discrepancy, used in [4], yields (7.2).

For an independent rigidity argument, $Q(f) = 1$ forces $\lambda = 1$. Every coordinate has variance and energy equal to one, so it attains the Poincaré inequality. Spectral calculus gives $Ax_i = x_i$ weakly. For the smooth density this says $\partial_i V = x_i$ for every i , whence $\mu = \gamma_n$. $\square$

The Wasserstein estimate is a consequence of an established Gaussian stability theorem, not a new stability theorem. It is consistent with [theorem 4.3](#): closeness in transport distance does not control the infimum of the Hessian in the far tails.

8 Low spectral packets and the KLS assertion

We return to the form setting of [section 2](#); no smooth potential, discrete spectrum, or attained gap is assumed. All measures in this section are isotropic and log-concave. Let P denote the spectral resolution of A . For a unit vector θ , put $\ell_\theta(x) = \langle x, \theta \rangle$ and define its spectral measure by

$$\rho_\theta(B) = \|P(B)\ell_\theta\|_2^2.$$

The constant projection is zero, and isotropy and the form energy give

$$\rho_\theta([\lambda, \infty)) = 1, \quad \int s d\rho_\theta(s) = 1. \quad (8.1)$$

The second identity needs only $\ell_\theta \in \text{Dom } A^{1/2}$, not $\ell_\theta \in \text{Dom } A$.

Lemma 8.1 (A low-frequency bound). *If $f \in \text{Ran } P((0, R])$ is nonzero, then*

$$Q_\mu(f) \leq R. \quad (8.2)$$

More generally, for $f \in \text{Dom } A$ of positive energy,

$$Q_\mu(f) \leq \frac{\|Af\|_2^2}{\mathcal{E}(f, f)}.$$

Proof. Use (2.3); on the spectral interval $(0, R]$, $s^2 \leq Rs$, so $\|Af\|_2^2 \leq R\mathcal{E}(f, f)$. $\square$

Theorem 8.2 (Packet equivalence). *The assertions and constants of [theorem C](#) hold. More precisely, if $C_P(\mu) \leq K$, then for every $T > 1$ and every unit vector θ ,*

$$f_\theta = P((0, T])\ell_\theta \quad (8.3)$$

is nonzero, belongs to $\text{Ran } P((0, TK\lambda])$, and satisfies

$$Q_\mu(f_\theta) \geq \lambda(1 - 1/T) \geq (1 - 1/T)/K. \quad (8.4)$$

Proof. Suppose first that some nonzero $f \in \text{Ran } P((0, L\lambda])$ has $Q(f) \geq q$. By [theorem 8.1](#), $q \leq L\lambda$, so $C_P(\mu) \leq L/q$.

Conversely, let $\lambda \geq 1/K$. From [\(8.1\)](#) and Markov's inequality,

$$\rho_\theta((T, \infty)) \leq 1/T, \quad \rho_\theta((0, T]) \geq 1 - 1/T > 0.$$

Thus $f_\theta \neq 0$. Its energy is

$$e_\theta := \mathcal{E}(f_\theta, f_\theta) = \int_{(0, T]} s \, d\rho_\theta(s) \geq \lambda(1 - 1/T). \quad (8.5)$$

Spectral orthogonality gives

$$\left\langle \int \nabla f_\theta \, d\mu, \theta \right\rangle = \mathcal{E}(f_\theta, \ell_\theta) = \mathcal{E}(f_\theta, f_\theta) = e_\theta.$$

Consequently $Q(f_\theta) \geq e_\theta$, proving [\(8.4\)](#). Finally $TK\lambda \geq T$, so this fixed absolute cutoff is contained in the required multiplicative spectral window. $\square$

The proof explains the role of a packet. The coordinate function has unit L^2 mass and unit spectral first moment. A bound on the gap prevents all of its useful low-frequency mass from lying arbitrarily far above the bottom. No corresponding argument singles out an individual eigenvalue.

8.1 A quasimode formulation

Corollary 8.3. *The universal Poincaré bound in [theorem C](#) is equivalent to the existence of universal constants $C < \infty$ and $q > 0$ such that every isotropic log-concave measure admits a mean-zero $f \in \text{Dom } A$, of positive energy, with*

$$\|Af\|_2^2 \leq C\lambda\mathcal{E}(f, f), \quad Q_\mu(f) \geq q. \quad (8.6)$$

The latter assertion implies $C_P(\mu) \leq C/q$.

Proof. The implication to a gap bound follows from [theorem 8.1](#). For the converse, use the functions in [theorem 8.2](#), for which $\|Af\|_2^2 \leq T\mathcal{E}(f, f) \leq TK\lambda\mathcal{E}(f, f)$. $\square$

Here “quasimode” refers to the second-moment spectral condition [\(8.6\)](#), not to a small residual $\|(A - \lambda)f\|_2$. These are different notions and should not be interchanged.

8.2 Aggregate spectral mass and sum rules

For a Borel set $B \subset (0, \infty)$ define the coordinate energy measure

$$M_\mu(B) = \sum_{i=1}^n \|A^{1/2}P(B)x_i\|_2^2. \quad (8.7)$$

This is basis-independent, and $M_\mu((0, \infty)) = n$.

Proposition 8.4. *The universal Poincaré bound is equivalent to the existence of universal $L < \infty$ and $q > 0$ such that*

$$M_\mu((0, L\lambda]) \geq qn \quad (8.8)$$

for every isotropic log-concave measure. This condition implies $C_P(\mu) \leq L/q$. Conversely, $C_P(\mu) \leq K$ implies it with $L = TK$, $q = (1 - 1/T)/K$ for any $T > 1$.

Proof. Since $\sum_i \|x_i\|_2^2 = n$,

$$M_\mu((0, L\lambda]) \leq L\lambda \sum_i \|P((0, L\lambda])x_i\|_2^2 \leq L\lambda n.$$

This proves the first implication. Conversely, sum (8.5) over the coordinate directions and use $TK\lambda \geq T$. $\square$

Proposition 8.5 (Discrete sum rules). *Suppose in addition that A has a complete real orthonormal eigenbasis $1, f_1, f_2, \dots$ with positive eigenvalues λ_j , counted with multiplicity. Then*

$$\sum_{j \geq 1} Q_\mu(f_j) = n, \quad \sum_{j \geq 1} \frac{Q_\mu(f_j)}{\lambda_j} = n. \quad (8.9)$$

Moreover, $M_\mu(B) = \sum_{\lambda_j \in B} Q_\mu(f_j)$.

Proof. The coordinate identity and $\mathcal{E}(f_j, f_j) = \lambda_j$ yield

$$Q_\mu(f_j) = \lambda_j \sum_{i=1}^n |\langle x_i, f_j \rangle|^2.$$

Summing first with the weight 1 and then with $1/\lambda_j$ gives, respectively, $\sum_i \mathcal{E}(x_i, x_i) = n$ and $\sum_i \|x_i\|_2^2 = n$ by the spectral theorem. All sums have nonnegative terms. Restriction to B proves the final identity. $\square$

Thus preferred-direction mass cannot disappear from the whole spectrum, even though it can disappear asymptotically from its first eigenspace. The invariant spectral measure (8.7), rather than a choice of eigenbasis inside a multiple eigenvalue, is the appropriate object for this statement.

9 Uniform measures on convex bodies: a remaining step

The preceding counterexamples concern weighted diffusions. The qualitative convex-body result described in the introduction remains unaffected, and the note does not establish a convex-body counterexample to a dimension-free quantitative estimate. We record a construction that clarifies what such an extension would require.

Proposition 9.1 (An isotropic body with a prescribed axial marginal). *Let $I = [-b, b]$ and let w be a smooth, positive, even probability density on I , with $\int_I t^2 w(t) dt = 1$. Suppose*

$$(\log w)'' \leq -c < 0 \quad \text{on } I.$$

For all sufficiently large integers m , put

$$K_m = \{(t, z) \in I \times \mathbb{R}^m : |z| \leq R_m(t)\}, \quad R_m(t) = c_m w(t)^{1/m}, \quad c_m^2 = \frac{m+2}{\int_I w^{1+2/m} dt}. \quad (9.1)$$

Then K_m is a convex unconditional body. Its uniform probability measure is isotropic, and its t -marginal is exactly $w(t) dt$.

Proof. Writing $\ell = \log w$, we have

$$(w^{1/m})'' = w^{1/m} \left(\frac{\ell''}{m} + \frac{(\ell')^2}{m^2} \right).$$

This is nonpositive whenever $m \geq \sup_I (\ell')^2 / (-\ell'')$. Hence R_m is concave and positive, which makes K_m convex. Evenness and Euclidean symmetry in z give unconditionality. The fiber volume is a constant times $R_m(t)^m$, which is a constant times $w(t)$, so the axial marginal is w .

Conditional on t , a uniform point in the m -ball of radius $R_m(t)$ has covariance $R_m(t)^2 \text{Id} / (m+2)$. Thus each transverse coordinate has variance

$$\frac{c_m^2}{m+2} \int_I w(t)^{1+2/m} dt = 1.$$

All cross-covariances and means vanish by symmetry, and the axial variance is one by assumption. $\square$

For fixed w , R_m is of order $\sqrt{m}$, while R'_m is of order $m^{-1/2}$. This suggests a limiting axial diffusion with invariant density w , coupled to transverse modes on a scale comparable to one. An axial test function $F(t, z) = f(t)$ has exactly the weighted Rayleigh quotient

$$\frac{\int_{K_m} |\nabla F|^2}{\int_{K_m} F^2} = \frac{\int_I |f'|^2 w dt}{\int_I f^2 w dt} \quad \text{when } \int_I f w dt = 0.$$

This proves an upper bound on the first Neumann eigenvalue, but it does not identify the first eigenspace: a function depending only on t does not in general satisfy the Neumann boundary condition on the sloping side of K_m .

Question 9.2. *Assume that the one-dimensional Neumann diffusion with invariant density w has a simple first eigenvalue $\lambda_w < 1$, with normalized eigenfunction f_w . Is it true that, for the bodies (9.1),*

$$\lambda_1(K_m) \longrightarrow \lambda_w, \quad Q_{K_m}(F_m) \longrightarrow Q_w(f_w)$$

for normalized first eigenfunctions F_m , with control of the entire first eigenspace if multiplicity occurs?

To use the question for a counterexample one would first approximate a standardized hyperbolic model by such compactly supported densities, retaining its isolated first mode, and then prove the asserted high-dimensional spectral limit. One would also need to control any smoothing of the body's boundary if a smooth-body statement is desired. Neither the convergence in [theorem 9.2](#) nor the full passage to convex bodies is proved in this note. Marginal convergence or Rayleigh upper bounds alone cannot supply the missing conclusion about mean gradients.

A The rest of the hyperbolic spectrum

For reference, the factorization used in [theorem 4.1](#) determines all the discrete eigenvalues. This also puts the endpoint $a = 2$ in context.

Proposition A.1. *For $a > 0$, the essential spectrum of A_a is $[a^2/4, \infty)$. Its eigenvalues below that interval are simple and are exactly*

$$\lambda_k = k(a - k), \quad k \in \mathbb{Z}_{\geq 0}, \quad k < a/2. \quad (\text{A.1})$$

There is no eigenfunction at the threshold $a^2/4$. In particular, for $0 < a \leq 2$ the spectral gap is $a^2/4$ and is not attained.

Proof. The essential-spectrum statement follows from [\(4.5\)](#). For $0 < a \leq 2$, the partner operator is

$$D_a D_a^* = -\frac{d^2}{dx^2} + \frac{a^2}{4} + \frac{a(2-a)}{4} \operatorname{sech}^2 x \geq \frac{a^2}{4}.$$

It has no eigenfunction at $a^2/4$: equality of its quadratic form with that value would force a square-integrable eigenfunction to have zero derivative. Consequently H_a has only its ground-state eigenvalue below the threshold and no eigenvalue at the threshold. For $a > 2$, the nonzero spectrum below the threshold is the spectrum of H_{a-2} shifted by $a - 1$, by [\(4.6\)](#). Iteration terminates at a parameter in $(0, 2]$. It gives exactly [\(A.1\)](#), with no threshold eigenvalue. Simplicity follows at each step from the one-dimensional ground state and the intertwining factorization. $\square$

Up to a nonzero constant, a corresponding diffusion eigenfunction can be written as

$$f_k(x) = \cosh^k x P_k^{(a/2-k, a/2-k)}(\tanh x), \quad (\text{A.2})$$

where $P_k^{(\alpha, \alpha)}$ is the Jacobi polynomial. Substitution in the Jacobi differential equation verifies the eigenvalue, and $a > 2k$ is its integrability condition. Its parity is that of k . Thus higher even modes have zero mean derivative, while the first mode is odd. This parity observation concerns higher eigenfunctions only and is not the mechanism of [theorem 4.1](#).

The logarithmic Gamma transform [\(5.6\)](#) has the same list $k(a - k)$ of bound-state eigenvalues and the same threshold, but it is a different Schrödinger potential. Its eigenfunctions are expressed through Laguerre rather than Jacobi polynomials; see [\[11\]](#). The two models should not be conflated merely because their bound-state energies agree.

References

- [1] D. Bakry, I. Gentil, and M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, Grundlehren der mathematischen Wissenschaften, vol. 348, Springer, Cham, 2014.
- [2] S. G. Bobkov, Isoperimetric and analytic inequalities for log-concave probability measures, *Ann. Probab.* **27** (1999), no. 4, 1903–1921.
- [3] H. J. Brascamp and E. H. Lieb, On extensions of the Brunn–Minkowski and Prékopa–Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation, *J. Funct. Anal.* **22** (1976), no. 4, 366–389. doi:10.1016/0022-1236(76)90004-5.

- [4] T. A. Courtade, M. Fathi, and A. Pananjady, Existence of Stein kernels under a spectral gap, and discrepancy bounds, *Ann. Inst. Henri Poincaré Probab. Stat.* **55** (2019), no. 2, 777–790. doi:10.1214/18-AIHP898. arXiv:1703.07707.
- [5] R. Kannan, L. Lovász, and M. Simonovits, Isoperimetric problems for convex bodies and a localization lemma, *Discrete Comput. Geom.* **13** (1995), 541–559.
- [6] B. Klartag, A Berry–Esseen type inequality for convex bodies with an unconditional basis, *Probab. Theory Related Fields* **145** (2009), 1–33. arXiv:0705.0832.
- [7] B. Klartag, Logarithmic bounds for isoperimetry and slices of convex sets, *Ars Inven. Anal.* (2023), Paper No. 4, 17 pp. doi:10.15781/jsjy-0b06. arXiv:2303.14938.
- [8] G. V. Livshyts, *MATH-8803: Topics Course in Concentration of Measure Phenomena and Convexity*, Georgia Institute of Technology, Fall 2023 lecture notes, version of July 16, 2024. <https://glivshyts6.math.gatech.edu/Conc-lecture-notes.pdf>.
- [9] G. Pöschl and E. Teller, Bemerkungen zur Quantenmechanik des anharmonischen Oszillators, *Z. Phys.* **83** (1933), 143–151. doi:10.1007/BF01331132.
- [10] G. Teschl, *Mathematical Methods in Quantum Mechanics: With Applications to Schrödinger Operators*, second edition, Graduate Studies in Mathematics, vol. 157, American Mathematical Society, Providence, RI, 2014. <https://www.mat.univie.ac.at/~gerald/ftp/book-schroe/schroe2.pdf>.
- [11] P. Zhang, Morse potential, contour integrals, and Asian options, arXiv:1010.3820, 2010.