HW 7: POLYNOMIALS, LINEAR ALGEBRA

1. Prove that the zeros of the polynomial $P(z) = z^7 + 7z^4 + 4z + 1$ lie inside the disk of radius 2 centered at the origin.

2. Let A and B be 2×2 matrices with real entries satisfying $(AB - BA)^n = I_2$ for some positive integer n. Prove that n is even and $(AB - BA)^4 = I_2$.

3. There are given $2n + 1$ real numbers, $n \geq 1$, with the property that whenever one of them is removed, the remaining $2n$ can be split into two sets of n elements that have the same sum of elements. Prove that all the numbers are equal.

4. Let A, B be 2×2 matrices with integer entries, such that $AB = BA$ and $\det B = 1$. Prove that if $\det(A^3 + B^3) = 1$, then $A^2 = 0$.

5. Let p be a prime integer. Prove that the determinant of the matrix

$$\begin{pmatrix} x & y & z \\ x^p & y^p & z^p \\ x^{p^2} & y^{p^2} & z^{p^2} \end{pmatrix}$$

is congruent modulo p to a product of polynomials of the form $ax + by + cz$, where a, b, c are integers. (We say two integer polynomials are congruent modulo p if corresponding coefficients are congruent modulo p.)

6. Find a nonzero polynomial $P(x, y)$ such that $P([a], [2a]) = 0$ for all real numbers a. (Note: $[v]$ is the greatest integer less than or equal to v.)

7. Show that the curve $x^3 + 3xy + y^3 = 1$ contains only one set of three distinct points, A, B, and C, which are vertices of an equilateral triangle, and find its area.