Convex bodies of constant width with exponential illumination number

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(joint work with Andrii Arman and Andrii Bondarenko)

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The smallest known n with $b(n) > n + 1$ is $n = 64$.
Bondarenko (2014): $b(65) > 83$, Jenrich (2014): $b(64) > 70$.

Asymptotic upper bound on $b(n)$

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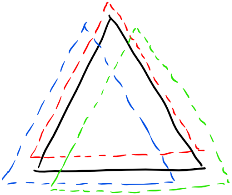

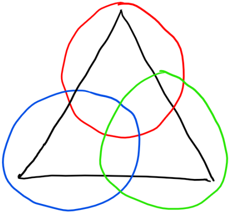

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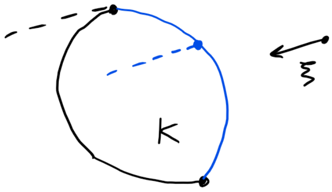
Bourgain and Lindenstrauss (1989): $1.0645^n \leq g(n) \leq \left(\sqrt{\frac{3}{2}} + o(1)\right)^n$.

Illumination and covering

Let K be a convex body in $\mathbb{E}^n$. A point $x \in \partial K$ is illuminated by a direction $\xi \in \mathbb{S}^{n-1}$ if the ray $\{x + \xi t : t \geq 0\}$ intersects $\mathrm{int}(K)$.



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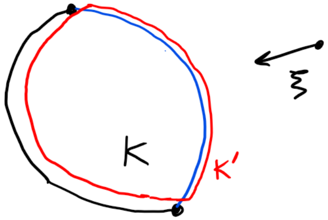

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Levi-Hadwiger-Gohberg-Markus's conjecture: $I(K) = h(K) \leq 2^n$
with equality iff K is an affine copy of a cube.

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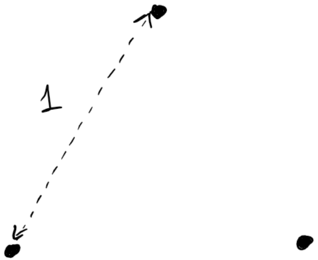

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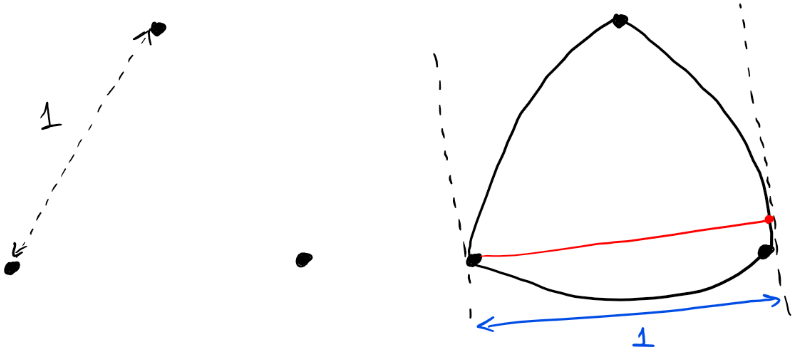

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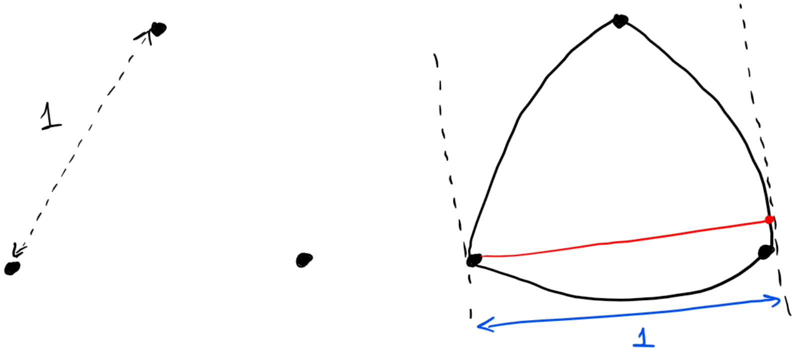

Therefore, it suffices to consider only bodies of constant width when computing the Borsuk's number $b(n)$.

Schramm's upper bound on Borsuk's number

Define

$$h(n) := \sup\{h(K) = I(K) : K \text{ is a convex body of constant width in } \mathbb{E}^n\}.$$

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Kalai (2015) asked: does there exist $C > 1$ with $h(n) \geq C^n$ for large n?

Main result

We answer the question of Kalai in the affirmative.

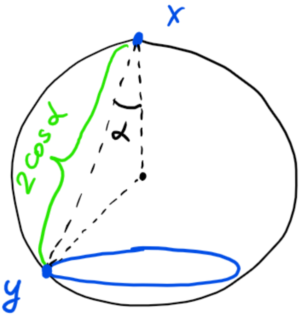
Theorem 1

$$h(n) \geq \frac{c}{\sqrt{n}\log n}\left(\frac{1}{\cos(\pi/14)}\right)^n$$

Main geometric ingredient

For fixed $x \in \mathbb{S}^{n-1}$ and $0 < \alpha \leq \pi/6$ define

$$Q(x,\alpha) := \{x\} \cup \{y \in \mathbb{S}^{n-1} : \|x - y\| = 2\cos\alpha\}.$$



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Suppose $0 < \alpha \leq \pi/6$, K is a convex body in $\mathbb{E}^n$ s.t. $\operatorname{diam} K = 2\cos\alpha$ and for some $x \in \mathbb{S}^{n-1}$ we have $Q(x, \alpha) \subset K$. Then $x \in \partial K$ and any direction $\xi \in \mathbb{S}^{d-1}$ illuminating x satisfies $\xi \in C(-x, \frac{\pi}{2} - \alpha)$.

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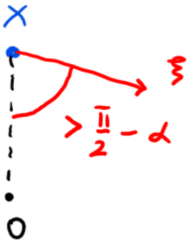

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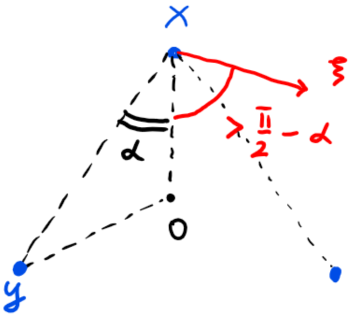

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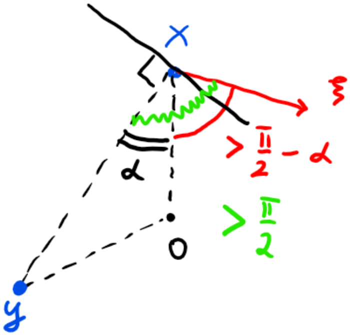

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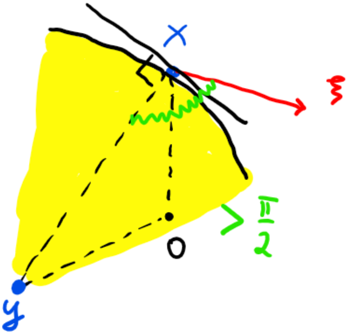

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Lemma 2

Suppose $0 < \alpha \leq \pi/6$ *and* $X \subset \mathbb{S}^{n-1}$.

(i) *If* $\theta(x, y) \leq \pi - 2\alpha$ *for all* $x, y \in X$, *then* $\operatorname{diam} X \leq 2 \cos \alpha$.

(ii) *If* $4\alpha \leq \theta(x, y) \leq \pi - 6\alpha$ *for all distinct* $x, y \in X$, *then* $\operatorname{diam} \mathcal{W}(X) \leq 2 \cos \alpha$.

Thin spherical codes

Lemma 3

Suppose $0 < \varphi < \frac{\pi}{2}$. *Then for any sufficiently large* n *there exists a collection* $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{c\sqrt{n}}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$;

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If μ denotes the spherical probability measure on $\mathbb{S}^{n-1}$, then up to a constant factor $\mu(C(x_i, \varphi))$ behaves like $\frac{(\sin\varphi)^n}{\sqrt{n}}$ for large n.

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Proof outline: Sample an appropriately selected number of uniformly i.i.d. points from $\mathbb{S}^{n-1}$. By Böröczky and Wintsche (2003), which is the adaptation of the ideas of Erdős and Rogers (1961/62) to $\mathbb{S}^{n-1}$, the resulting set Y satisfies (b) with high probability.

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Certain probabilistic arguments show that some points that may violate (a) can be removed from Y to obtain the desired $X \subset Y$.

Proof of the main result

Theorem 1

$$h(n) \geq \frac{c}{\sqrt{n}\log n}\left(\frac{1}{\cos(\pi/14)}\right)^n$$

Proof: Use Lemma 3 with $\varphi = \frac{6\pi}{14}$ to get a thinly spread $X \subset \mathbb{S}^{n-1}$.

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Glazyrin ($\geq$2023) noted that the base of the exponent $\frac{1}{\cos(\pi/14)} \approx 1.026$ can be improved to $\frac{1}{4}\sqrt{\frac{1}{6}(111-\sqrt{33})} \approx 1.047$ by a slight modification of the construction: choosing the bases of the cones from a concentric sphere of smaller radius.

New lower bound on $g(n)$

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Bourgain and Lindenstrauss (1989): $g(n) \geq 1.0645^n$

Theorem 2

$$g(n) \geq \frac{c}{\sqrt{n}\log n}\left(\frac{2}{\sqrt{3}}\right)^n \qquad \textit{(note that } \tfrac{2}{\sqrt{3}} \approx 1.1547\textit{)}$$

Proof: Use Lemma 3 with $\varphi = \frac{\pi}{3}$ to get a thinly spread $X \subset \mathbb{S}^{n-1}$.
By Lemma 2 (i) (separation lemma) with $\alpha = \frac{\pi}{6}$, $\operatorname{diam} X \leq 2\cos\frac{\pi}{6} = \sqrt{3}$.

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Any ball of diameter $\sqrt{3}$ intersects $\mathbb{S}^{n-1}$ by a cap of radius $\leq \varphi$, so by Lemma 3 (b) we need at least $\frac{c\sqrt{n}}{(\sin\varphi)^n}/(Cn\log n) = \frac{c'}{\sqrt{n}\log n}\left(\frac{2}{\sqrt{3}}\right)^n$ such caps to cover X.

Thin spherical codes

Denote $\mu(\varphi) := \mu(C(x, \varphi))$, $x \in \mathbb{S}^{n-1}$.

Theorem 3

There is n_0 such that for any $n \geq n_0$, $\psi \in (0, \frac{\pi}{2})$ and $\varphi \in (\frac{1}{n}, \frac{\pi}{2})$ there exists a collection $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{n-1}$ with $N \geq \min\{\frac{4n \log n}{\mu(\varphi)}, \frac{1}{8\mu(\psi)}\}$ such that

(a) $\psi \leq \theta(x_i, x_j) \leq \pi - \psi$ *for all* $i \neq j$*;*

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Lemma 3 is obtained when $\psi = \varphi$.

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• If $pM > \frac{1}{2}$, draw $T \subset Y$ selecting each point with probability $\frac{1}{2pM}$. Then $\mathbb{E}(|T| - |B(T)|) \geq \frac{1}{2p} - pM^2(\frac{1}{2pM})^2 = \frac{1}{4p} = \frac{1}{8\mu(\psi)}$.

Illumination of convex bodies close to ball

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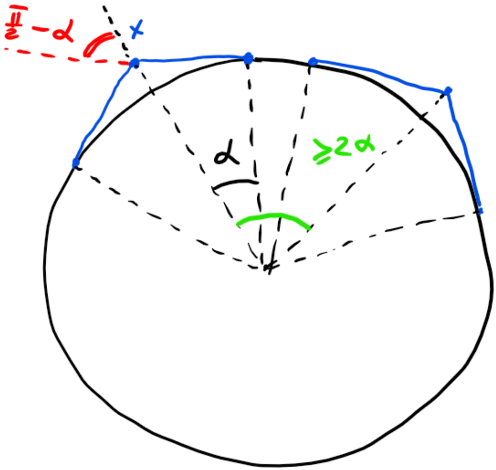
Construction: convex hull of a discrete subset of $D\mathbb{S}^{n-1}$ and $\mathbb{B}^n$.

Spiky ball



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For any fixed $1 < D < \frac{2}{\sqrt{3}}$ (≈ 1.1547) *and sufficiently large* n

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Illumination of bodies of constant width close to the ball

For $D > 1$ let $\mathcal{W}_D^n$ be the family of all convex bodies of *constant width* $K \subset \mathbb{E}^n$ such that

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For any fixed $1 < D < \frac{1}{2\cos(\pi/14)-1}$ (≈ 1.0528) *and sufficiently large* n

$$c\sqrt{n}\left(\frac{2D}{D+1}\right)^n \leq \sup_{K \in \mathcal{W}_D^n} I(K)$$

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If the inradius of K is 1 and the circumradius is D, then the width is $D+1$. Therefore, after rescaling, such a body would have constant width 1 and the diameter of the circumscribed sphere would be $\frac{2D}{D+1}$.

Covering by balls of smaller diameter

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Upper bounds in the last two theorems are achieved in a “universal” way: illumination directions and covering balls do not depend on K, only on D.

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Question

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Question

Is it true that $I(K) = g(K)$ for any K of constant width? If not, are $I(K)$ and $g(K)$ for constant width $K \subset \mathbb{E}^n$ equivalent up to a factor polynomial in n?