High-dimensional phenomena: a public lecture

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HIM, Bonn, Germany
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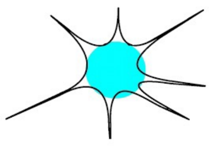
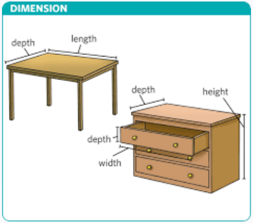
to Asia (my math circles teacher)



What is dimension?..

We live in a space where there is 3 dimensions:

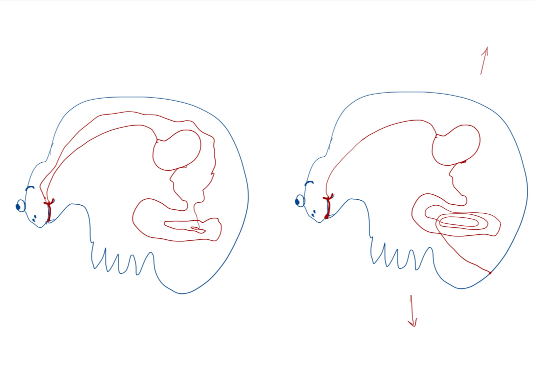


Can we imagine a 2-dimensional space? Think about shadows that have width and depth, but no height!

Fun fact about 2 dimensions

Fun fact (related to me by Asia, my math circles teacher)

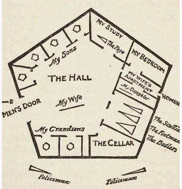
If a creature lived in the 2-dimensional space it would need to eat and defecate through the same opening...



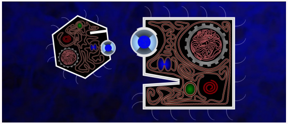
... or else it would fall apart!

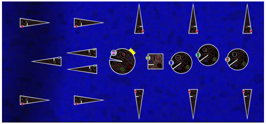
Edwin Abbott Abbott, *Flatland: a romance in many dimensions*, 1884

- “Flatland” is a book about life in two dimensions, narrated by a square. Characters are triangles, squares, polygons, circles and others



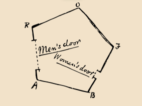
- Intelligence is somewhat proportional to the smallest angle; also, the more sides the wiser the creature
- Sons of a polygon with N sides have $N+1$ sides; hence the rule “respect and honor your children”
- Circles are the wisest, the land is ruled by the Chief Circle, priests are circles, while soldiers are the least witty and they are tall triangles



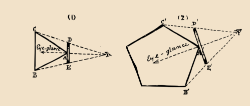


Edwin Abbott Abbott, *Flatland: a romance in many dimensions*, 1884

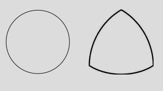
- Women are straight lines (intervals). Women are supposed to move side to side all the time in order to be seen. Men and women have separate entrance to a dwelling.

- All men are seen as intervals; their shape is inferred by the sense of feeling. Alternatively, one can understand the shape by looking at them in the "fog" which makes farther points appear dimmer



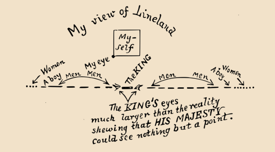
- According to Abbott, one could learn a shape by walking around it... but this is not accurate:



- Question: and what if the shapes have to be polygons, can we learn them by walking around them (and learning the lengths of all projections)?

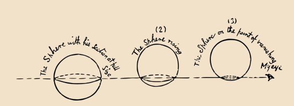
Edwin Abbott Abbott, *Flatland: a romance in many dimensions*, 1884

- The narrator (the square) gets the "gospel of the three dimensions". The night before that, he dreams of the... Lineland!



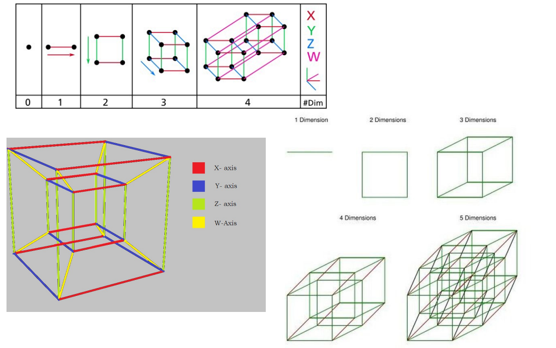
- In Lineland, all creatures are intervals, communicate using several voices; for coitus, no proximity is needed
- The square tries to explain to the Line King that he is 2-dimensional by vanishing from the Lineland:

- Later on, he gets explained 3D space by a sphere who uses the same trick:



Fun facts about 4 dimensions

Similarly, maybe we could try and understand 4 dimensions by applying our experience of looking down to 2 dimensions from our 3-dimensional world?



Imagine: if we (as 3-dimensional beings) lived in 4 dimensions, we would be able to touch one another's insides!

Fun facts about 4 dimensions

Also, if we (as 3-dimensional beings) lived in 4 dimensions, this is how a pair of shoes would look like:

Guess why?

Fun facts about 4 dimensions

Also, if we (as 3-dimensional beings) lived in 4 dimensions, this is how a pair of shoes would look like:

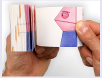
Guess why?

Just like we are able to move the left image (below) to the right using the freedom of 3 dimensions,

our feet would be identical in 4 dimensions, and hence we would be able to wear a left boot on a right foot!

How to make sense out of many dimensions?

Think about a flipbook:



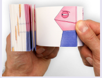
A collection of pictures in 2D makes a 3D book; when shuffled, it makes a little flat movie!

Similarly, when our space moves in time, it creates a 4D picture. *Time* can be viewed as a fourth dimension.

What about a 5-dimensional space?

How to make sense out of many dimensions?

Think about a flipbook:



A collection of pictures in 2D makes a 3D book; when shuffled, it makes a little flat movie!

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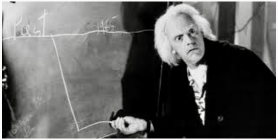
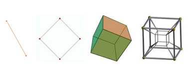
What about a 5-dimensional space?



Also a good book to read: Ted Chiang, "Anxiety is the dizziness of freedom."

Four-dimensional cube is called tesseract

Segment, square, cube... tesseract!

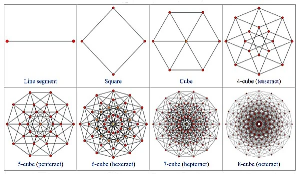


To understand tesseract (the 4-dimensional cube) imagine that the cube is falling at you, as in the picture above on the right.

In culture

- Robert Heinlein, “And he built a crooked house”, about a house built as a 3D unfolded tesseract which accidentally folds back during an earthquake
- Appears in the Marvel universe, as well as the movie “Interstellar”

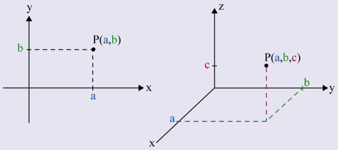
More dimensions (see also plug-ins):



Mathematically speaking...

Alternatively, one can imagine a 4D space by taking *density of materials* as the fourth parameter, in addition to *width, length and height*.

Mathematically, a point in 2D is a pair of Cartesian coordinates (a, b). A point in 3D is a triplet (a, b, c). A point in 4D is a quadruplet (a, b, c, d).

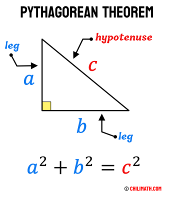


A large number of parameters or variables might be needed to describe a system. In a human population, they may include *height, weight, age, gender, number of languages spoken, eye color (indexed by a number), hair color,...*

Geometric viewpoint, stemming from our geometric intuition in 2D and 3D may help us study such complex high-dimensional systems.

Distances in high dimensions

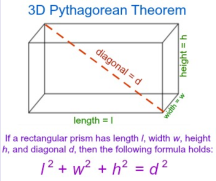
Pythagorean theorem in 2D



Pythagorean theorem in 3D and n dimensions

Similarly, in an n-dimensional space, distance from the origin to the point $x = (x_1, \dots, x_n)$ is

$$\sqrt{x_1^2 + \dots + x_n^2}.$$

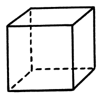


High-dimensional cube and ball

High-dimensional cube

A unit cube in an n-dimensional space $\mathbb{R}^n$ is defined as

$$B_\infty^n = \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \textit{for all } i,\ |x_i| \leq 1\right\}$$

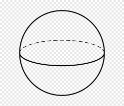


The sidelength of this cube is 2.

High-dimensional ball

A unit ball in an n-dimensional space $\mathbb{R}^n$ is defined as

$$B_2^n = \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x| = \sqrt{x_1^2 + \ldots + x_n^2} \leq 1\right\}$$



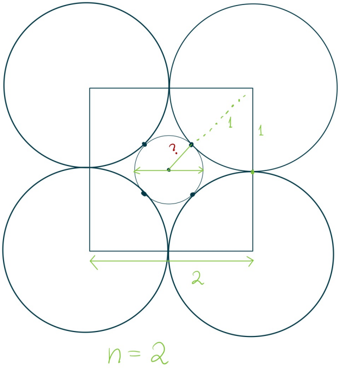
From now on we will think of the dimension

$$n = 1000000,$$

or, more generally, let n be a very large number! It turns out, higher dimensionality sometimes introduces greater *simplicity* rather than complexity.

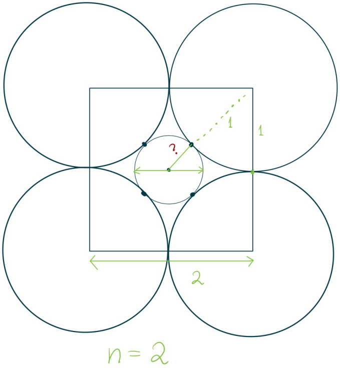
Weird high-dimensional phenomenon #1

What is the length of this interval when the dimension $n = 2$?



Weird high-dimensional phenomenon #1

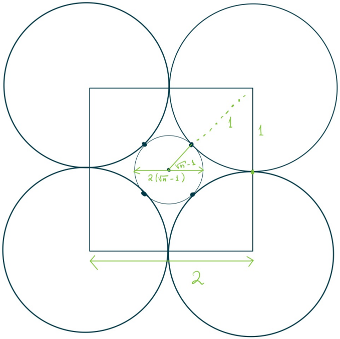
The length is $\sqrt{2} - 1 \approx 0.4$.



$? = \sqrt{2} - 1 \approx 0.4$

Weird high-dimensional phenomenon #1

And what if the dimension n is arbitrary?

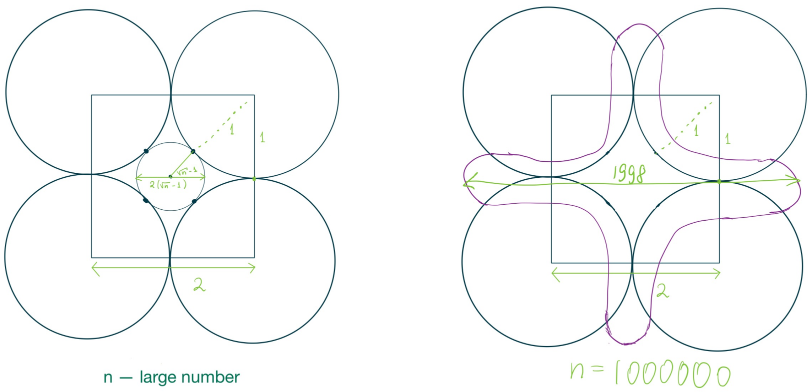


n — large number

RaDIUS = $\sqrt{n} - 1$

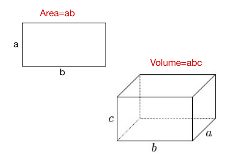
Weird high-dimensional phenomenon #1

The ball "spills out"!



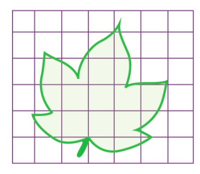
Area, volume, *n*-dimensional Lebesgue measure

Area of a rectangle and volume of a parallelepiped



Area of any set (Lebesgue measure)

Imagine a set on a very fine grid and count how many grid squares the set covers; then add their areas (which we know to find – see above). This gives an approximation of the area of the set.



The situation in higher dimensions is the same!

Homogeneity of the n-dimensional Lebesgue measure

Homogeneity property of volume

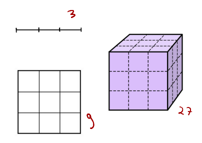
For a set A in $\mathbb{R}^n$, we have

$$Vol_n(5A) = 5^n Vol_n(A).$$

Here $5A = \{5x : x \in A\}$.

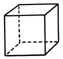
More generally, for $t > 0$,

$$Vol_n(tA) = t^n Vol_n(A).$$



Volume of the unit cube B_∞^n

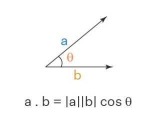
$$Vol_n(B_\infty^n) = 2^n.$$



Review of scalar products and projection

Let a and b be vectors, then the *scalar product:*

$$\langle a, b\rangle = |a| \cdot |b| \cdot \cos(a, b),$$



In coordinates, if $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$, we have

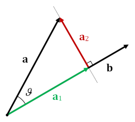
$$\langle a, b\rangle = a_1 b_1 + ... + a_n b_n.$$

Projection

If $|b| = 1$ then

$$\langle a, b\rangle = |a| \cos(a, b)$$

is the length of the projection of a onto b.



Law of Large numbers

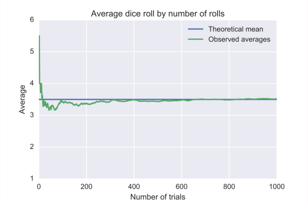
Law of Large Numbers

Let $X_1, \dots, X_n$ be independent random variables with $\mathbb{E}X_i = E$. Then

$$\frac{X_1 + \dots + X_n}{n} \longrightarrow_{n \to \infty} E$$

in the appropriate sense.

In other words, if you pull a ticket numbered 1, 2, 3 out of a hat, and average your numbers after 10000 trials, you will almost certainly get something very close to 2 (since $\frac{1+2+3}{3} = 2$).



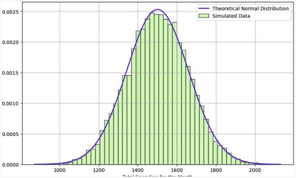
Central Limit Theorem

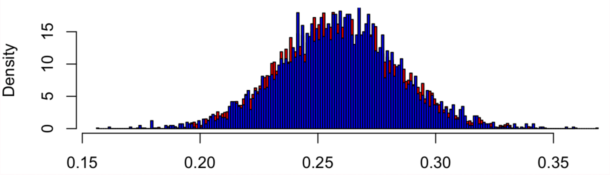
Central limit theorem

Let $X_1, \ldots, X_n$ be independent random variables with bounded variance and $\mathbb{E}X_i = E$. Then

$$\frac{X_1 + \ldots + X_n}{\sqrt{n}} \longrightarrow_{n\to\infty} \textit{Normal distribution}$$

in the appropriate sense.

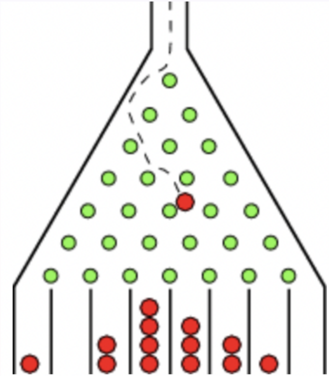




Central Limit Theorem

Galton's board

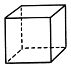
Imagine a bunch of beads falling down and randomly reflecting right or left at many levels. Each reflection is a ± 1 random variable X_i, and the position of the bead is the averaged sum of the X_i.



Geometric interpretation of the LLN and CLT

Central limit theorem

Let $X_1,\ldots,X_n$ be independent random variables uniform on $[-\frac{1}{2},\frac{1}{2}]$. Look at the vector $X=(X_1,\ldots,X_n)$. It is uniformly distributed in the cube $Q=[-\frac{1}{2},\frac{1}{2}]^n$.



Consider the random variable

$$\frac{X_1+\ldots+X_n}{\sqrt{n}}=\langle X,\theta\rangle,$$

where $\theta=\left(\frac{1}{\sqrt{n}},\ldots,\frac{1}{\sqrt{n}}\right)$. Note that

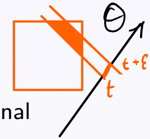
$$|\theta|=\sqrt{\theta_1^2+\ldots+\theta_n^2}=\sqrt{\frac{1}{n}+\ldots+\frac{1}{n}}=1,$$

and thus $\langle X,\theta\rangle$ is the projection of X onto the direction θ.

Geometric interpretation of the LLN and CLT

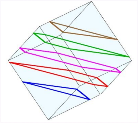
The density of the random variable $\langle X, \theta \rangle$ is

$$f_{\langle X,\theta \rangle}(t) \approx \frac{1}{\epsilon} P(\langle X, \theta \rangle \in [t, t+\epsilon]) \approx Vol_{n-1}(Q \cap H_t),$$



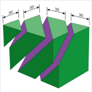
– the *section function of the cube*, where H_t is the $(n-1)-$dimensional hyperplane distance t from the origin. Thus, by the CLT,

$$Vol_{n-1}(Q \cap H_t) \longrightarrow_{n\to\infty} Normal\ distribution,$$



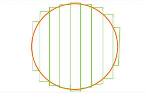
Also, by the LLN, there is a thin enough slab S orthogonal to $\theta = \left(\frac{1}{\sqrt{n}}, ..., \frac{1}{\sqrt{n}}\right)$ such that

$$Vol_n(S \cap Q) = 99\%\ of\ the\ cube.$$



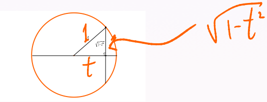
Volume of the n-dimensional ball

The volume of the unit ball B_2^n in $\mathbb{R}^n$ can be found using Fubbini's theorem: approximate the ball with the union of many thin ball-slabs B_i, write



$$Vol_n(B_2^n) \approx \sum Vol(B_i) \approx \int_{-1}^{1} Vol_{n-1}(B_t)dt,$$

where B_t is a unit ball in R^{n-1} of radius $\sqrt{1-t^2}$.



By homogeneity,

$$Vol_{n-1}(B_t) = Vol_{n-1}\left(\sqrt{1-t^2}B_2^{n-1}\right) = (1-t^2)^{\frac{n-1}{2}} Vol_{n-1}\left(B_2^{n-1}\right).$$

Therefore,

$$Vol_n(B_2^n) = Vol_{n-1}\left(B_2^{n-1}\right) \int_{-1}^{1} (1-t^2)^{\frac{n-1}{2}} dt.$$

Volume of the n-dimensional ball

$$Vol_n(B_2^n) = Vol_{n-1}\left(B_2^{n-1}\right)\int_{-1}^{1}(1-t^2)^{\frac{n-1}{2}}\,dt.$$

Letting

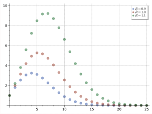
$$J_n = \int_{-1}^{1}(1-t^2)^{\frac{n-1}{2}}\,dt,$$

we see that

$$Vol_n(B_2^n) = J_n \cdot Vol_{n-1}\left(B_2^{n-1}\right) = J_n \cdot J_{n-1} \cdot Vol_{n-2}\left(B_2^{n-2}\right) = \ldots = J_n \cdot J_{n-1} \cdot \ldots \cdot J_1.$$

Dim n	1	2	3	4	5	6	7	8	9	10	11	12
$Vol_n(B_2^n)$	2	π	$\frac{4\pi}{3}$	$\frac{\pi^2}{2}$	$\frac{8\pi^2}{15}$	$\frac{\pi^3}{6}$	$\frac{16\pi^3}{105}$	$\frac{\pi^4}{24}$	$\frac{32\pi^4}{945}$	$\frac{\pi^5}{120}$	$\frac{64\pi^5}{10395}$	$\frac{\pi^6}{720}$
$\approx$	2	3.14	4.19	4.93	5.27	5.17	4.73	4.06	3.30	2.55	1.88	1.34

Volume of unit ball B_2^n decays rapidly to zero as the dimension $n \to \infty$:



Volume of the n-dimensional ball

$$\frac{Vol_n(B_2^n)}{Vol_n(B_2^{n-1})} = J_n = \int_{-1}^{1} (1-t^2)^{\frac{n-1}{2}}\, dt \approx \frac{\sqrt{2\pi}}{\sqrt{n}}$$

as $n \to \infty$. More precisely,

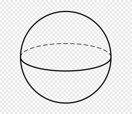
$$Vol_n(B_2^n) \approx \left(\frac{2\pi e}{n}\right)^{\frac{n}{2}},$$

decays to zero very fast! Consequently,

$$Vol_n(R_n B_2^n) = 1$$

when $R_n \approx 0.242\sqrt{n}$.

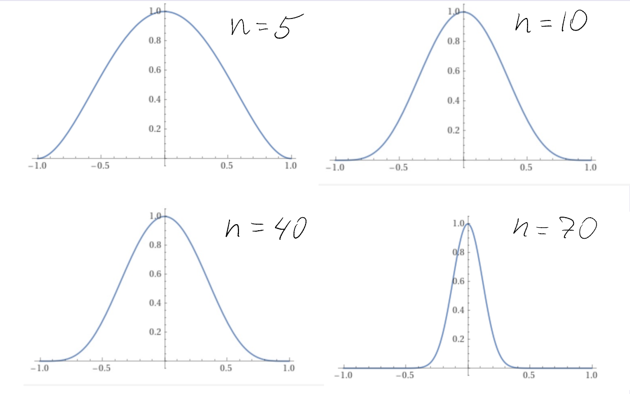
In dimension 1000000, the ball of volume 1 has radius approximately 242.



Volume of the n-dimensional ball

Focus on

$$\frac{Vol_n(B_2^n)}{Vol_n(B_2^{n-1})} = J_n = \int_{-1}^{1} (1-t^2)^{\frac{n-1}{2}}\, dt \approx \frac{\sqrt{2\pi}}{\sqrt{n}}.$$

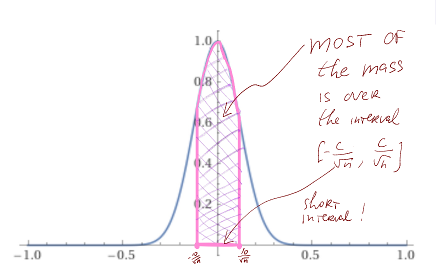


The mass under the graph becomes more and more *concentrated* around the peak!

Volume of the n-dimensional ball

Suppose for concreteness that $n \geq 1000$.

$$\frac{\sqrt{2\pi}}{\sqrt{n}} \approx \int_{-1}^{1} (1-t^2)^{\frac{n-1}{2}}\, dt \geq \int_{-\frac{10}{\sqrt{n-1}}}^{\frac{10}{\sqrt{n-1}}} (1-t^2)^{\frac{n-1}{2}}\, dt \geq \frac{20\left(1-\frac{100}{n-1}\right)^{\frac{n-1}{2}}}{\sqrt{n-1}} = \frac{c}{\sqrt{n}}.$$



Thus

$$\int_{-1}^{1} (1-t^2)^{\frac{n-1}{2}}\, dt \leq C \int_{-\frac{10}{\sqrt{n-1}}}^{\frac{10}{\sqrt{n-1}}} (1-t^2)^{\frac{n-1}{2}}\, dt.$$

Concentration in the n-dimensional ball

Suppose for concreteness that $n \geq 1000$

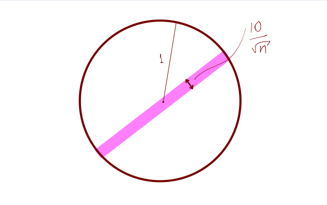
Let

$$B = \left\{ x \in B_2^n : |x_1| \leq \frac{10}{\sqrt{n}} \right\}.$$

Then

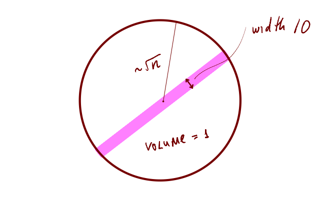
$$Vol_n(B) = Vol_{n-1}(B_2^n) \int_{-\frac{10}{\sqrt{n-1}}}^{\frac{10}{\sqrt{n-1}}} (1-t^2)^{\frac{n-1}{2}} dt \geq$$

$$0.99 Vol_{n-1}(B_2^n) \int_{-1}^{1} (1-t^2)^{\frac{n-1}{2}} dt = 0.99 Vol_n(B_2^n).$$



Concentration in the n-dimensional ball

99% of the volume of the Euclidean ball of radius 1 in $\mathbb{R}^n$ comes from the strip of width of order $\frac{1}{\sqrt{n}}$! For instance, in dimension 1000000, most of the volume of the unit ball comes from the strip of width 0.001 around the equator!



One could also look at the $\sqrt{n}-$scaled picture: 99% of the mass of the ball of volume 1 (whose radius is about $\sqrt{n}$) comes from a strip of constant width.
This is an analogue of the law of large numbers – but without the independence!

Concentration in the n-dimensional ball

Surprisingly, mass concentrates strongly near every equator! So which picture below is a high-dimensional ball?..

Concentration in the n-dimensional ball

This might make one think that the ball is especially heavy near the center... But this is not true!

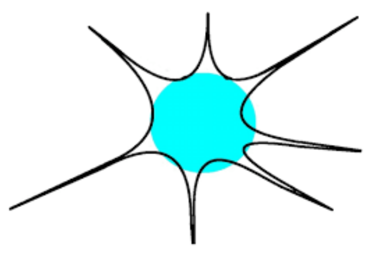
$$Shell = \left\{x : |x| \in [1 - \frac{10}{n}, 1]\right\}.$$

Then

$$Vol_n(Shell) = Vol_n(B_2^n) - Vol_n\left(\left(1 - \frac{10}{n}\right) B_2^n\right) =$$

$$\left(1 - \left(1 - \frac{10}{n}\right)^n\right) Vol_n(B_2^n) \approx 0.99 \cdot Vol_n(B_2^n).$$

Most of the volume of the ball is $\frac{1}{n}$—near the boundary! This is a classical depiction of the high-dimensional ball:

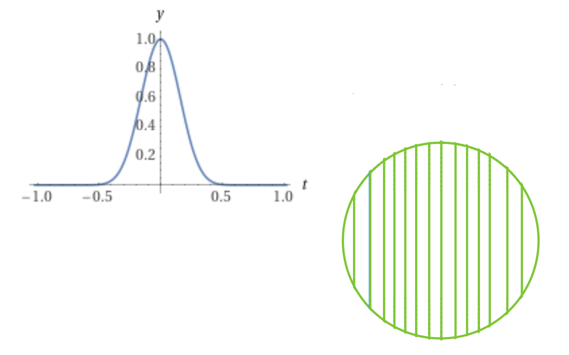


Central Limit Theorem for the ball

In analogy with the geometric interpretation of the Central Limit Theorem (which we saw earlier with the cube), we notice:

CLT for the ball

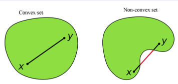
The function $|B_2^n \cap H_t|$ (where H_t is a subspace distance t from the origin) is close to a Gaussian, if the dimension is large.



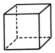
Central Limit Theorem for convex sets

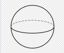
Convex set

A set K is *convex* if for any $x, y \in K$, the line connecting x and y is inside K.



Examples of convex sets:

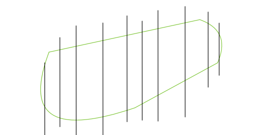


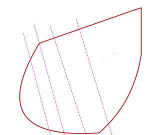


A version of the Central Limit Theorem is true for all convex sets K!

Theorem (Klartag 2007)

For *any* convex set K in $\mathbb{R}^n$ there is some direction in which $Vol_{n-1}(K \cap H_t)$ "looks like" the standard normal distribution.





Bourgain's slicing problem

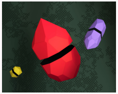
Law of Large numbers for convex sets

If X is uniform in a K, a convex set in $\mathbb{R}^n$ of volume 1, and if the thinnest strip containing 50% of the volume of K is orthogonal to $(1, 1, ..., 1)$, then does $\frac{X_1 + ... + X_n}{n} \to const$?

The answer is yes (follows from the Theorem below), but one may even ask a harder question: what is the biggest possible width of the thinnest strip containing 50% of the volume of K?

Equivalent question: Bourgain's slicing problem, 1984

Does convex K of volume 1 have an $(n-1)$-dimensional section of "area" at least $1/100$?



Theorem (Klartag, Lehec 2024)

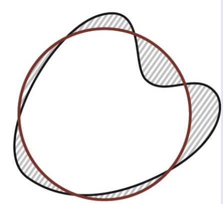
Yes!

Isoperimetric inequality

A Roman soldier is given a rope of fixed length in order to surround a plot of land. What shape should he make in order to maximize the area?

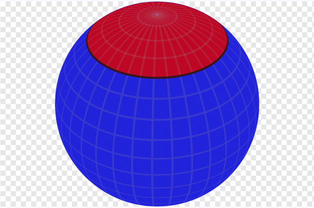
The isoperimetric inequality (ancient romans and greeks, Jacob Steiner 1834, 20th century...)

If $A \subset \mathbb{R}^n$ has fixed volume then its "perimeter" is minimized when A is a ball.

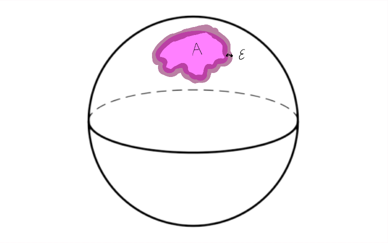


Spherical isoperimetric inequality

Analogously, denoting the sphere in $\mathbb{R}^n$ by $\mathbb{S}^{n-1}$, the perimeter of $A \subset \mathbb{S}^{n-1}$ of given "area" is smallest when A is a spherical cap.

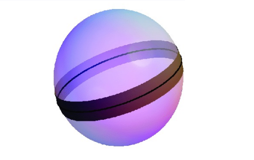


From this one can infer that also the "ϵ-thickening" of A (consisting of the points on the sphere which are ϵ-close to A) is the smallest, among A of fixed "area", if A is a spherical cap.



Concentration in the high-dimensional sphere

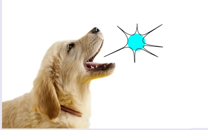
Like for the high-dimensional ball, there is *measure concentration for the sphere*: 99% of the mass of the sphere is $\frac{c}{\sqrt{n}}$-near any equator!



In fact, more is true! Suppose $\epsilon \approx \frac{100}{\sqrt{n}}$.

A large enough set eats the sphere!

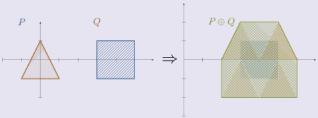
Let $A \subset \mathbb{S}^{n-1}$ be a set which takes up 50% of the sphere. Let A_ϵ be a "thickening" of A. Then A_ϵ takes up 99% of the whole sphere.

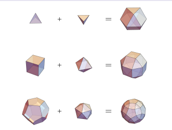


Brunn-Minkowski inequality

Minkowski sum of sets $K, L \subset \mathbb{R}^n$

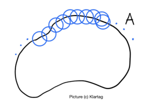
$$K + L = \{x + y : x \in K, y \in L\}$$





Perimeter of a set A

$$Perim(A) \approx \frac{Vol_n(A + \epsilon B_2^n) - Vol_n(A)}{\epsilon}$$



The Brunn-Minkowski inequality

$$Vol_n(K + L)^{\frac{1}{n}} \geq Vol_n(K)^{\frac{1}{n}} + Vol_n(L)^{\frac{1}{n}}$$

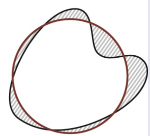
Isoperimetry via Brunn-Minkowski inequality

perimeter of the ball is n times the volume

$$Perim(B_2^n) \approx \frac{1}{\epsilon}\left(Vol_n((1+\epsilon)B_2^n) - Vol_n(B_2^n)\right) = \frac{(1+\epsilon)^n - 1}{\epsilon} Vol_n(B_2^n) = n Vol_n(B_2^n).$$

The isoperimetric inequality (revisited)

For (nice) sets K with $Vol_n(K) = Vol_n(B_2^n)$, one has $Perim(K) \geq Perim(B_2^n)$.



Proof (via the Brunn-Minkowski inequality)

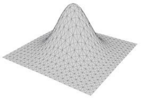
$$Perim(K) \approx \frac{Vol_n(K + \epsilon B_2^n) - Vol_n(K)}{\epsilon} \geq$$

$$\frac{\left(Vol_n(K)^{\frac{1}{n}} + \epsilon Vol_n(B_2^n)^{\frac{1}{n}}\right)^n - Vol_n(K)}{\epsilon} \approx n Vol_n(K)^{\frac{n-1}{n}} Vol_n(B_2^n)^{\frac{1}{n}}.$$

and when $Vol_n(K) = Vol_n(B_2^n)$ we get $n Vol_n(B_2^n)$ (=perimeter of the ball.) $\square$

I feel obliged to also mention some of my own theorems:)

The multivariate normal distribution $d\gamma(x) = \frac{1}{\sqrt{2\pi}^n} e^{-\frac{|x|^2}{2}} dx$



Theorem (Kolesnikov, Livshyts 2018)

For a pair of convex sets K, L containing the origin, one has

$$\gamma\left(\frac{K+L}{2}\right)^{\frac{1}{2n}} \geq \frac{\gamma(K)^{\frac{1}{2n}} + \gamma(L)^{\frac{1}{2n}}}{2}$$

Extended by Eskenazis, Moschidis; Cordero-Erausquin, Rotem; convexity dropped by Aishwarya, Rotem.

Theorem (Livshyts 2021)

For a pair of symmetric convex sets K, L, even log-concave measure μ on $\mathbb{R}^n$ (whose density's logarithm is concave), with $c_n = 1/poly(n)$, we have

$$\mu\left(\frac{K+L}{2}\right)^{c_n} \geq \frac{\mu(K)^{c_n} + \mu(L)^{c_n}}{2}.$$

Thanks for your attention!

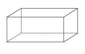
(drawing by Itay B.)

Appendix: proof of the Brunn-Minkowksi inequality

$$Vol_n(K+L)^{\frac{1}{n}} \geq Vol_n(K)^{\frac{1}{n}} + Vol_n(L)^{\frac{1}{n}}$$

Proof of the Brunn-Minkowski inequality.

STEP 1: When K and L are coordinate boxes given by $K = \prod_{i=1}^n [0, a_i]$ and $L = \prod_{i=1}^n [0, b_i]$, we have $K + L = \prod_{i=1}^n [0, a_i + b_i]$.



Then the Brunn-Minkowski inequality amounts to

$$\left(\prod_{i=1}^n (a_i + b_i)\right)^{\frac{1}{n}} \geq \left(\prod_{i=1}^n a_i\right)^{\frac{1}{n}} + \left(\prod_{i=1}^n b_i\right)^{\frac{1}{n}},$$

or equivalently,

$$1 \geq \left(\prod_{i=1}^n \frac{a_i}{a_i + b_i}\right)^{\frac{1}{n}} + \left(\prod_{i=1}^n \frac{b_i}{a_i + b_i}\right)^{\frac{1}{n}},$$

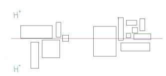
and this follows from the arithmetic-geometric mean inequality.

Appendix: proof of the Brunn-Minkowksi inequality

Proof of the Brunn-Minkowski inequality.

STEP 2: For volume, K and L can be approximated by a union of boxes:

Argue by induction in the total number of boxes, call it N. Base case for $N=1$ was handled in Step 1. After shifting K as necessary, let H be a hyperplane splitting them into K^+ and K^- and L^+ and L^- so that $\frac{Vol_n(K^+)}{Vol_n(K)} = \frac{Vol_n(L^+)}{Vol_n(L)} = t$, and such that the total number of boxes is less than N on both sides.



By inductional assumption,

$$Vol_n(K+L) \geq Vol_n(K^+ + L^+) + Vol_n(K^- + L^-) \geq$$

$$\left(Vol_n(K^+)^{\frac{1}{n}} + Vol_n(L^+)^{\frac{1}{n}}\right)^n + \left(Vol_n(K^-)^{\frac{1}{n}} + Vol_n(L^-)^{\frac{1}{n}}\right)^n =$$

$$(t+1-t)\left(Vol_n(K)^{\frac{1}{n}} + Vol_n(L)^{\frac{1}{n}}\right)^n . \square$$