

# The null set of a polytope, and the Pompeiu property for polytopes

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Another restatement of the problem (easy) is:

Is it true that the integral of  $f$  over  $B$ , as well as integrals of  $f$  over all rigid motions of  $B$ , uniquely determine the function  $f$ ?



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BST proved that in  $\mathbb{R}^2$ , all Lipschitz curves ‘with at least one corner’ have the Pompeiu property.



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But our emphasis is on the zero set of the Fourier transform of a polytope  $\mathcal{P}$  - also called the null set of  $\mathcal{P}$ .



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$$\hat{1}_{\Delta}(\xi) = \left( \frac{1}{2\pi i} \right)^2 \left( \frac{1}{\xi_1 \xi_2} + \frac{be^{-2\pi i a \xi_1}}{(a\xi_1 - b\xi_2)\xi_1} + \frac{ae^{-2\pi i b \xi_2}}{(-a\xi_1 + b\xi_2)\xi_2} \right),$$



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(We are ignoring here a complication inherent in such a formula for general polytopes: the triangulation of the vertex tangent cones)



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(Note: it is tricky to visualize  $S_{\mathbb{C}}^{d-1}$ , even when  $d = 2$ , because even in this case we have a 2-dimensional, unbounded manifold sitting in  $\mathbb{R}^4$ , as one can easily check.)



Fact. The null set of a polytope gives a lot of information about the **combinatorics** of the polytope. In particular, it also gives us a necessary and sufficient condition for tiling and multi-tiling.



# Harmonic analysis lemma for tilings in terms of the null set of $\mathcal{P}$

**Lemma.** (M. Kolountzakis)

A convex polytope  $P$  admits a  $k$ -tiling of  $\mathbb{R}^d$  by translations with the lattice  $\mathcal{L}$  if and only if both of the following conditions are true:

$$(a) \quad \hat{1}_P(m) = 0, \text{ for all nonzero vectors } m \in \mathcal{L}^*$$

$$(b) \quad k = \frac{\text{vol} P}{|\det \mathcal{L}|}$$



Theorem. (Brown, Shreiber, and Taylor, 1973)



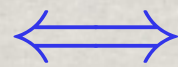
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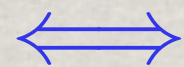
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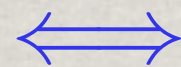


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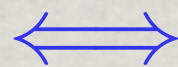
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In other words, Pompeiu's problem is equivalent to the claim that the null set  $N(P)$  does not contain any of the complex algebraic varieties  $S^{d-1}(r)$ .



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Known: In dimension 2, it is known to be true, but it is open in all higher dimensions.



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There are also **finite** analogues of the Pompeiu problem.



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But even for  $d = 2$  and for an arbitrary 5-point set, it is not completely known.



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We may recall that the explicit FT of a polytope is:

$$\int_{\mathcal{P}} e^{-2\pi i \langle u, z \rangle} du = \sum_{v \in V} \frac{e^{-2\pi i \langle v, z \rangle}}{(2\pi i)^d} \sum_{j=1}^M \frac{\det K_j}{\prod_{k=1}^d \langle w_{j,k}, z \rangle},$$

Suppose we parametrize a circle in Euclidean space by  $z(t) : [0, 1] \rightarrow \mathbb{R}^d$ .

We suppose, to the contrary, that the null set of  $\mathcal{P}$  **does contain** a circle.

We can massage the vanishing criterion  $0 = \hat{1}_{\mathcal{P}}(z(t))$  of the FT of a polytope, (given explicitly by the theorem above) into:

$$0 = \sum_{v \in V(\mathcal{P})} p_v(t) e^{2\pi i \langle v, z(t) \rangle},$$

where  $p_v(t)$  is an explicitly given trigonometric polynomial in  $t$ .



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$$J_n(x) := \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!k!} \left(\frac{x}{2}\right)^{2k}.$$



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an over-determined identity that is satisfied by  $r$ , for all  $n \in \mathbb{Z}$ .

Considering simple asymptotic values of the Bessel functions, for large  $n$ , we arrive at a contradiction.



Thank you