The null set of a polytope, and the Pompeiu property for polytopes

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Another restatement of the problem (easy) is:

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BST proved that in $\mathbb{R}^2$, all Lipschitz curves ‘with at least one corner’ have the Pompeiu property.

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But our emphasis is on the zero set of the Fourier transform of a polytope $\mathcal{P}$ - also called the null set of $\mathcal{P}$.

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(We are ignoring here a complication inherent in such a formula for general polytopes: the triangulation of the vertex tangent cones)

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(Note: it is tricky to visualize $S_{\mathbb{C}}^{d-1}$, even when $d = 2$, because even in this case we have a 2-dimensional, unbounded manifold sitting in $\mathbb{R}^4$, as one can easily check.)

Fact. The null set of a polytope gives a lot of information about the **combinatorics** of the polytope. In particular, it also gives us a necessary and sufficient condition for tiling and multi-tiling.

Harmonic analysis lemma for tilings in terms of the null set of $\mathcal{P}$

Lemma. (M. Kolountzakis)

A convex polytope P admits a k-tiling of $\mathbb{R}^d$ by translations with the lattice $\mathcal{L}$ if and only if both of the following conditions are true:

(a) $\hat{1}_P(m) = 0$, for all nonzero vectors $m \in \mathcal{L}^*$

(b) $k = \frac{\text{vol} P}{|det \mathcal{L}|}$

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In other words, Pompeiu's problem is equivalent to the claim that the null set $N(P)$ does not contain any of the complex algebraic varieties $S^{d-1}(r)$.

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Known: In dimension 2, it is known to be true, but it is open in all higher dimensions.

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There are also **finite** analogues of the Pompeiu problem.

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But even for $d = 2$ and for an arbitrary 5-point set, it is not completely known.

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where $p_v(t)$ is an explicitly given trigonometric polynomial in t.

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After some gymnastics with sums, we get a vanishing criterion that has the shape:

$$0 = \sum_{v \in V(\mathcal{P})} e^{-in\phi_v} \sum_{k=-N}^{N} c_{v,k}\, J_{n-k}(rr_v) i^k e^{ik\phi_v},$$

an over-determined identity that is satisfied by r, for all $n \in \mathbb{Z}$.

Considering simple asympotic values of the Bessel functions, for large n, we arrive at a contradiction.

Thank you