Magnetic Brunn-Minkowski and Borell-Brascamp-Lieb inequalities on Riemannian manifolds

AGA Seminar, September 2024

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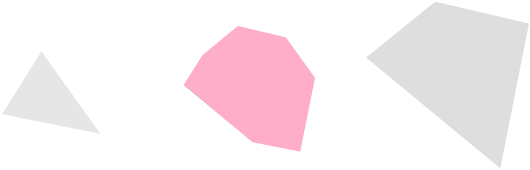
The Brunn-Minkowski inequality

$A_0, A_1 \subseteq \mathbb{R}^n, \ 0 < \lambda < 1.$

$$A_\lambda := (1-\lambda)A_0 + \lambda A_1 := \{(1-\lambda)a_0 + \lambda a_1 \mid a_0 \in A_0,\ a_1 \in A_1\}.$$



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Theorem (The Brunn-Minkowski inequality)

$A_0, A_1 \subseteq \mathbb{R}^n$ Borel, nonempty, $0 < \lambda < 1$,

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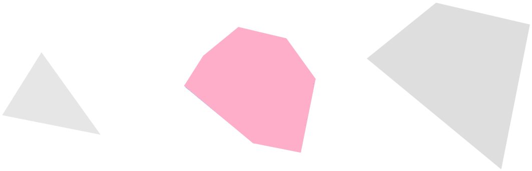

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Theorem (The Prékopa-Leindler inequality)

$0 < \lambda < 1$, $f_0, f_\lambda, f_1 : \mathbb{R}^n \to [0,\infty)$ integrable such that

$$f_\lambda((1-\lambda)x_0 + \lambda x_1) \geqslant f_0(x_0)^{1-\lambda} f_1(x_1)^\lambda \qquad \text{for every } x_0, x_1 \in \mathbb{R}^n.$$

$$\Longrightarrow \qquad \int_{\mathbb{R}^n} f_\lambda(x)dx \geqslant \left(\int_{\mathbb{R}^n} f_0(x)dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} f_1(x)dx\right)^{\lambda}.$$

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(M, g) Riemannian manifold, $A_0, A_1 \subseteq M$, $0 < \lambda < 1$.

$$A_\lambda := \left\{ \gamma(\lambda) \;\middle|\; \begin{array}{c} \gamma : [0,1] \to M \text{ constant speed minimizing} \\ \text{geodesic}, \quad \gamma(0) \in A_0, \quad \gamma(1) \in A_1. \end{array} \right\}$$

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Theorem (Cordero-Erausquin, McCann & Schmuckenschläger '01, Sturm '06)

(M, g) complete n-dim Riemannian manifold, $A_0, A_1 \subseteq M$ Borel, nonempty, $0 < \lambda < 1$, $\mathrm{Ric}_g \geqslant 0 \implies$

$$\mathrm{Vol}_g(A_\lambda)^{1/n} \geqslant (1-\lambda)\mathrm{Vol}_g(A_0)^{1/n} + \lambda \mathrm{Vol}_g(A_1)^{1/n}.$$

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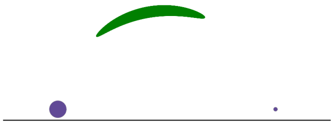

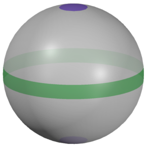

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- $\Longleftarrow$ Magnabosco, Portinale, Rossi '22.

The Borell-Brascamp-Lieb inequality

$$\mathbf{M}_q(a,b;\lambda) = \begin{cases} \left((1-\lambda)a^q + \lambda b^q\right)^{1/q} & q \in \mathbb{R}\setminus\{0\} \\ a^{1-\lambda}b^{\lambda} & q = 0 \\ \max\{a,b\} & q = +\infty \\ \min\{a,b\} & q = -\infty \end{cases}$$

Theorem (Cordero-Erausquin, McCann, Schmuckenschläger '01)

(M,g) complete n-dim Riemannian manifold, $q \in [-1/n, \infty]$, $0 < \lambda < 1$, $f_0, f_\lambda, f_1 : M \to [0,\infty)$ integrable such that

$$f_\lambda(\gamma(\lambda)) \geqslant \mathbf{M}_q\left(f_0(\gamma(0)), f_1(\gamma(1)); \lambda\right) \qquad \text{for every min. geo. } \gamma : [0,1] \to M.$$

If $\mathrm{Ric}_g \geqslant 0$ then:

$$\int_M f_\lambda d\mathrm{Vol}_g \geqslant \mathbf{M}_{\frac{q}{1+qN}}\left(\int_M f_0 d\mathrm{Vol}_g, \int_M f_1 d\mathrm{Vol}_g; \lambda\right).$$

(In Euclidean space due to Henstock & Macbeath '53, Borell '75, Brascamp & Lieb '76).

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Finsler manifolds (Ohta '09), Metric measure spaces (Bacher '10), Heisenberg groups (Balogha, Kristályb, Sipos '16)...

Horocyclic Brunn-Minkowski inequality

$\mathbf{H}^2$:= the hyperbolic plane. Let $A_0, A_1 \subseteq \mathbf{H}^2$.

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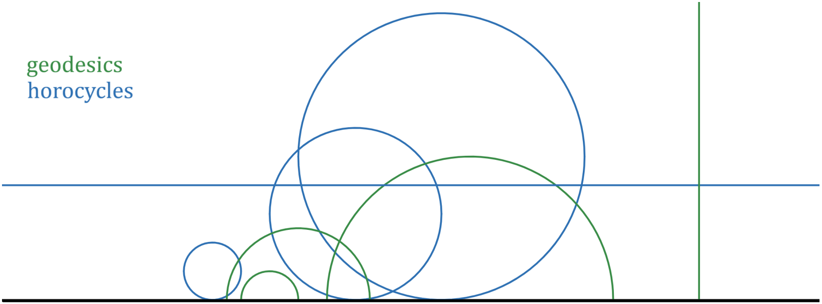

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Theorem (A., Klartag '22)

For every $A_0, A_1 \subseteq \mathbf{H}^2$ Borel, nonempty, and $0 < \lambda < 1$,

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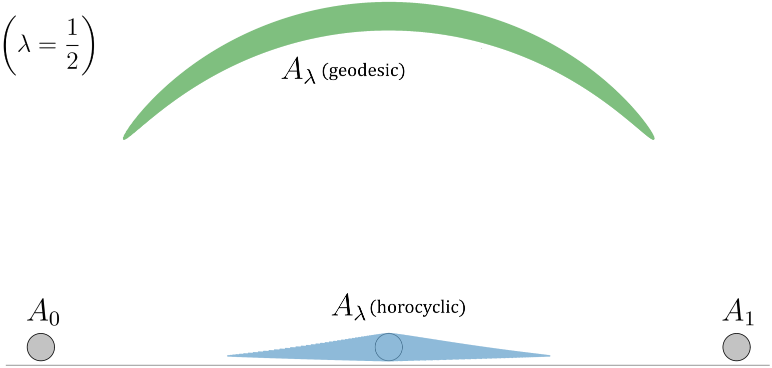

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When is it true that

$$\mu(A_\lambda) \geqslant \mathbf{M}_q(\mu(A_0), \mu(A_1); \lambda)$$

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- *On the curvature and heat flow on Hamiltonian systems*, Ohta '14

Theorem (A. '23)

Let (M, g) be a Riemannian surface (i.e. $\dim M = 2$).
$\Gamma \approx$ collection of constant-speed curves such that each $x, y \in M$ are joined by a unique curve $\gamma \in \Gamma$ (+ assumptions).

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TFAE:

1. For every pair of Borel, nonempty subsets $A_0, A_1 \subseteq M$ and every $0 < \lambda < 1$,

$$\mathrm{Vol}_g(A_\lambda)^{1/2} \geqslant (1-\lambda) \cdot \mathrm{Vol}_g(A_0)^{1/2} + \lambda \cdot \mathrm{Vol}_g(A_1)^{1/2},$$

2. There exists a smooth function $\kappa : M \to \mathbb{R}$ such that

$$\Gamma = \left\{ \text{solutions to the ODE} \quad \nabla_{\dot\gamma}\dot\gamma = \kappa \cdot |\dot\gamma| \cdot \dot\gamma^\perp \right\},$$

and moreover $K + \kappa^2 - |\nabla\kappa| \geqslant 0$, where K is the Gauss curvature of g.

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Integral curves of the Hamiltonian flow on T^*M with the Hamiltonian g and the canonical symplectic form ω;	Integral curves of the Hamiltonian flow on T^*M with the Hamiltonian g and the symplectic form $\omega_\Omega := \omega + \pi^*\Omega$.

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where J is the complex structure.

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Gouda '97, Grognet '99, Wojtkowski '00, Adachi '11, Assenza '23.

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In particular, if $\mathrm{mRic} \geqslant 0$ then J_t is $\frac{1}{n-1}$-concave (hence log-concave) in t.

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- **Properness**: For every compact $A \subseteq M$ there exists a compact $\tilde{A} \supseteq A$ such that every minimizing magnetic geodesic with endpoints in A is contained in $\tilde{A}$.

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$A_0, A_1 \subseteq M, 0 < \lambda < 1.$

$$A_\lambda := \left\{ \gamma(\lambda) \;\middle|\; \begin{array}{c} \gamma : [0,1] \to M \text{ constant speed minimizing} \\ \text{magnetic geodesic,} \quad \gamma(0) \in A_0, \quad \gamma(1) \in A_1 \end{array} \right\}.$$

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$\mathrm{mRic} \geqslant 0 \implies \forall\, A_0, A_1 \subseteq M$ Borel, nonempty, $0 < \lambda < 1$,

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$\mathrm{mRic} \geqslant K \in \mathbb{R} \implies$ "distorted Magnetic Brunn-Minkowski".

Example: complex hyperbolic space

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The unit ball $M = \{z \in \mathbb{C}^n \mid |z| < 1\}$ with the symplectic form and metric

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Weighted Borell-Brascamp-Lieb

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Theorem (Coredero-Erqusquin, McCann, Schmuckenschläger '01)

- (M, g) complete n-dimensional Riemannian manifold.
- μ measure on M with a smooth positive density.
- $N \in (n, \infty]$, $q \in [-1/N, \infty]$, $K \in \mathbb{R}$, $0 < \lambda < 1$.
- $f_0, f_\lambda, f_1 : M \to [0, \infty)$ integrable such that for every mininimizing geodesic $\gamma : [0, 1] \to M$,

$$f_\lambda(\gamma(\lambda)) \geqslant \mathbf{M}_q \left(\frac{f_0(\gamma(0))}{\beta_{1-\lambda}^{K,N}(\ell)}, \frac{f_1(\gamma(1))}{\beta_{\lambda}^{K,N}(\ell)}; \lambda \right), \qquad \text{where } \ell = \mathrm{Len}[\gamma].$$

Then, if $\mathrm{Ric}_{\mu,N} \geqslant K$ then

$$\int_M f_\lambda d\mu \geqslant \mathbf{M}_{\frac{q}{1+qN}} \left(\int_M f_0 d\mu, \int_M f_1 d\mu; \lambda \right).$$

Weighted magnetic Borell-Brascamp-Lieb

Theorem (A. '24)

- (M, g, η) satisfying the assumptions.
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Proof outline

Magnetic needle decomposition

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Evans-Gangbo '99, ... , Klartag '14, Ohta '15. Magnetic geodesics are reparametrizations of geodesics of the Finsler metric $F(v) := |v| - \eta(v)$.

Magnetic needle decomposition

Lemma

For ν-a.e. $\alpha \in \mathscr{A}$ and every $N \in (-\infty, \infty] \setminus [1, n]$, the density $d\mu_\alpha / d\mathcal{H}^1 =: e^{-\psi_\alpha}$ of the needle μ_α satisfies

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More applications of the magnetic needle decomposition

Theorem (Magnetic Poincaré inequality)

Assume that $\mathrm{mRic}_{\mu,\infty} \geqslant 0$. Then for every C^1 function $f : M \to \mathbb{R}$,

$$\int_M f\, d\mu = 0 \qquad \Longrightarrow \qquad \int_M f^2 d\mu \leqslant \frac{D^2}{\pi^2} \int_M |\nabla f|^2 d\mu,$$

where D is the length of the longest minimizing magnetic geodesic in M.

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- Log-Sobolev inequalities
- Isoperimetric inequalities (Lévy-Gromov, Bakry-Ledoux)
- ...

Lagrangians and Hamiltonians

A *Lagrangian* is a function $L : TM \to \mathbb{R}$. Associated to each Lagrangian is the Hamiltonian

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The *magnetic Lagrangian* and its associated Hamiltonian are

$$L(v) = \frac{|v|^2 + 1}{2} - \eta(v), \quad v \in TM; \qquad H(p) = \frac{|p + \eta|^2 - 1}{2}, \quad p \in T^*M.$$

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Recall $\mathcal{S} := \mathrm{Len} - \int \eta$.

Lemma

Let $T > 0$. Let $\gamma : [0, T] \to M$ be a piecewise-C^1 curve. Then

$$\mathcal{S}[\gamma] \leqslant \int_0^T L(\dot{\gamma}(t))dt,$$

with equality if and only if γ is parametrized by arclength.

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Lemma (Contreras-Iturriaga-Paternain-Paternain '98)

Assume that there exists $\varepsilon_0 > 0$ such that $\int_\gamma \eta \leqslant (1 - \varepsilon_0) \cdot \text{Len}[\gamma]$ for every closed curve γ on M. Then there exist $\varepsilon_1 > 0$ and a smooth function $\vartheta : M \to \mathbb{R}$ which is a strict Hamilton-Jacobi subsolution, i.e.

$$|d\vartheta + \eta| \leqslant 1 - \varepsilon_1. \tag{1}$$

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Theorem (Fathi '03, Bernard-Zavidovique '13, Klartag '14)

Let L be a Tonelli or Finsler Lagrangian on a manifold M, and let u be an L-dominated function. For every $\varepsilon > 0$ there exists a $C^{1,1}$ function $u_\varepsilon : M \to \mathbb{R}$ such that

$$u_\varepsilon \equiv u \qquad \text{and} \qquad du_\varepsilon \equiv du \qquad \text{on } \mathcal{S}_\varepsilon,$$

and for a.e. $x \in \mathcal{S}_\varepsilon$, the Hessian $\mathrm{Hess}\, u_\varepsilon$ exists and varies differentiably (and if L is Finsler, smoothly) on the calibrated curve through x.

Future study

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3. Non-magnetic sprays. (In dimension two, among constant-speed sprays, only magnetic sprays can satisfy Brunn-Minkowski).

Thank You!