Godbersen's conjecture for locally anti-blocking bodies

Shay Sadovsky

Tel Aviv University

Online AGA Seminar

Grünbaum gave the following classical definition for a measure of symmetry of convex sets, that it is a function, $f : \mathcal{K}^n \to [0, 1]$, such that

1. For any centrally-symmetric K, $f(K) = 1$.
2. f is invariant under affine transformations.
3. f is continuous.

A well known example of a measure of symmetry is the following quantity,

$$f(K) = \frac{2^n \mathrm{Vol}(K)}{\mathrm{Vol}(K - K)}.$$

which clearly satisfies (1)-(3).
Rogers and Shephard showed that for K convex,

$$\mathrm{Vol}(K - K) \leq \binom{2n}{n} \mathrm{Vol}(K),$$

with equality attained only for simplices, thus showing that simplices are also minimizers of $f(\cdot)$.

Recalling the definition of mixed volume,

$$\mathrm{Vol}(K - \lambda K) = \sum_{j=0}^{n} \binom{n}{j} \lambda^j V(K[j], -K[n-j]),$$

it is natural to ask whether the Rogers-Shephard inequality still holds for this polynomial,

$$(?) \qquad \mathrm{Vol}(K - \lambda K) \le \sum_{j=0}^{n} \binom{n}{j}^2 \lambda^j \mathrm{Vol}(K).$$

Conjecture (Godbersen's conjecture ('38), (Makai Jr. '74))

For any convex body $K \subset \mathbb{R}^n$ and $1 \leq j \leq n-1$,

$$V(K[j], -K[n-j]) \leq \binom{n}{j} \mathrm{Vol}(K)$$

with equality attained only for simplices.

As more motivation for the conjecture, note that the following quanitiy is also a measure of symmetry,

$$g_j(K) = \frac{\mathrm{Vol}(K)}{V(K[j], -K[n-j])}.$$

Godbersen's conjecture implies that this measure of symmetry is also minimized only by simplices.

It also implies the following conjecture about the λ-difference body

$$\frac{\mathrm{Vol}(D_\lambda K)}{\mathrm{Vol}(K)} \leq \frac{\mathrm{Vol}(D_\lambda \Delta)}{\mathrm{Vol}(\Delta)_n},$$

where $\Delta_n = \mathrm{conv}\left(\{0\} \cup \{e_i\}_{i=1}^n\right)$ and $D_\lambda K = (1-\lambda)K + \lambda(-K)$.

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Previous work on Godebersen's conjecture has yeided the following partial results:

- (Artstein-Avidan, Einhorn, Florentin, and Ostrover)

$$V(K[j], -K[n-j]) \leq \frac{n^n}{j^j (n-j)^{(n-j)}} \mathrm{Vol}(K) \approx \binom{n}{j} \mathrm{Vol}(K) \sqrt{2\pi \frac{j(n-j)}{n}}.$$

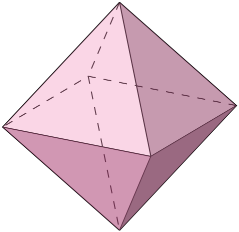
- (Artstein-Avidan)

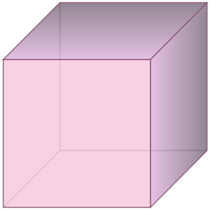
$$\sum_{j=0}^{n} \lambda^j (1-\lambda)^{n-j} V(K[j], -K[n-k]) \leq \mathrm{Vol}(K).$$

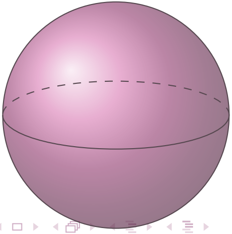
- (Artstein-Avidan, S, and Sanyal) The conjecture holds for a class called 'anti-blocking bodies' or 'convex corners'.

Definition

- A convex body $K \subseteq \mathbb{R}^n$ is called **unconditional** if $(x_1, \dots, x_n) \in K$ implies $(\pm x_1, \dots, \pm x_n) \in K$.
- A convex body K_+ is called **anti-blocking** if it is of the form $K_+ = K \cap \mathbb{R}^n_+$ when K is unconditional.
- A convex body $K \subseteq \mathbb{R}^n$ is called **locally anti-blocking** if, for any coordinate hyperplane $E_J^c = \mathrm{sp}\{e_j\}_{j \in J}$, $J \subset [n]$ (where $\{e_i\}_{i=1}^n$ is the standard basis), one has $P_E K = K \cap E$.
    - Alternatively, they can be defined as bodies that are anti-blocking in each orthant.
    - **Denote** the anti-blocking in each orthant by K_σ for $\sigma \in \{-1, 1\}^n$.



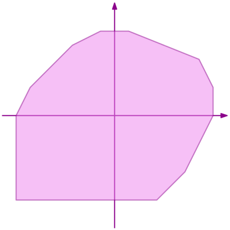


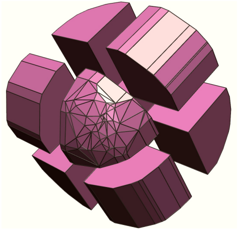


Locally anti-blocking bodies support a few conjectures already: Mahler's Conjecture (Fradelizi and Meyer) and Kalai's 3^d conjecture (Sanyal and Winter).

Theorem

Godbersen's conjecture holds for locally anti-blocking bodies. Among these, equality holds only for simplices.





For the equality cases we also need this neat observation:

Lemma (Mixed volume of simplices)

Let $\alpha_1, \ldots, \alpha_n \geq 0$ *and let* $K = \mathrm{conv}(0, \alpha_1 e_1, \ldots, \alpha_n e_n)$. *Then, for any* $0 \leq j \leq n$,

$$V_n(K[j], \Delta_n[n-j]) = \frac{1}{n!} \max \left\{ \prod_{i \in I} \alpha_i : |I| = j \right\}.$$

This allows one to compute the mixed volume of any two aligned simplices.

Proof of theorem

The proof of the inequality needs:

1. (AA-S-S) Mixed volume formula for two anti-blocking bodies.
2. Decomposition lemma of mixed volume for two locally anti-blocking bodies.
3. Behavior of coordinate projections of locally anti-blocking bodies.
4. (AA-S-S) A ‘mixed’ reverse Kleitman inequality.

(1) Mixed volume formula for two anti-blocking bodies

Let $K, K' \subseteq \mathbb{R}^n_+$ be anti-blocking bodies, let $0 \le j \le n$. Then

$$V_n(K[j], -K'[n-j]) = \binom{n}{j}^{-1} \sum_{\substack{E \text{ a } j\text{-dim.} \\ \text{coord. hyperplane}}} \mathrm{Vol}_j(P_E K) \cdot \mathrm{Vol}_{n-j}(P_{E^\perp} K').$$

Proof idea: A geometric decompostion of the sum $K - K'$ into a disjoint union

$$\bigcup_{\substack{E \text{ a coord.} \\ \text{hyperplane}}} P_E(K) \times P_{E^\perp}(-K').$$

Look at λK and compare coefficients.

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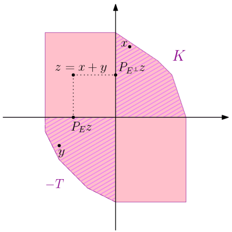

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(2) Decomposition lemma of mixed volume

Let $K, K' \subseteq \mathbb{R}^n$ be locally anti-blocking bodies. Then,

$$\mathrm{Vol}(K + K') = \sum_{\sigma \in \{-1,1\}^n} \mathrm{Vol}(K_\sigma + K'_\sigma)$$

and in fact,

$$V_n(K[j], K'[n-j]) = \sum_{\sigma \in \{-1,1\}^n} V_n(K_\sigma[j], K'_\sigma[n-j]).$$

Proof idea: Show that again there is a disjoint decomposition,

$$K + K' = \bigcup_\sigma K_\sigma + K'_\sigma.$$

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(3) Coordinate projections of locally anti-blocking bodiess

Let $K \subset \mathbb{R}^n$ be locally anti-blocking, and let $E := \mathrm{sp}\{e_i : i \in I\} \subset \mathbb{R}^n$ for some $I \subset [n]$. Let $\tau, \sigma \in \{-1, +1\}^n$ be two orthant signs, such that $\tau|_I = \sigma|_I$. Then,

$$P_E K_\sigma = P_E K_\tau.$$

Proof idea: Use the fact that for anti-blocking bodies (and locally anti-blocking) $P_E K = K \cap E$ and use that $K_\sigma \cap E = K_\tau \cap E$.

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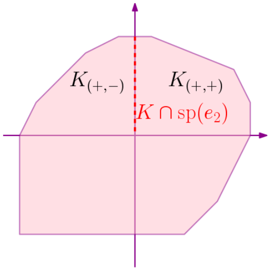

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(4) A 'mixed' reverse Kleitman inequality

Given two anti-blocking bodies, $K, T \subseteq \mathbb{R}^n_+$,

$$V_n(K[j], T[n-j]) \;\leq\; V_n(K[j], -T[n-j]).$$

Proof idea: Apply Steiner symmetrization along coordinates to K and $-T$ until they are unconditional, and show that the mixed volume only decreases along the symmetrization (Shephard's theorem for shadow systems).

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Proof

Let K be locally anti-blocking, $K = \cup_\sigma K_\sigma$ with each K_σ anti-blocking in $\sigma\mathbb{R}^n_+$.
Using the decomposition lemma (2),

$$\begin{aligned} V_n(K[j], -K[n-j]) &= \sum_{\sigma\in\{-1,1\}^n} V_n(K_\sigma[j], (-K)_\sigma[n-j]) \\ &= \sum_{\sigma\in\{-1,1\}^n} V_n(K_\sigma[j], -(K_{-\sigma})[n-j]) \\ &\leq \sum_{\sigma\in\{-1,1\}^n} V_n(K_\sigma[j], (K_{-\sigma})[n-j]). \qquad (\star) \end{aligned}$$

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Using the decomposition lemma (2), since $(-K)_\sigma = -(K_{-\sigma})$ and using the mixed reverse Kleitman inequality (4), we get

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Using our formula for mixed volumes of anti-blocking bodies (1) we see that

$$\sum_{\sigma\in\{-1,1\}^n} V_n(K_\sigma[j],(K_{-\sigma})[n-j])$$

$$= \sum_{\sigma\in\{-1,1\}^n} \binom{n}{j}^{-1} \sum_{\substack{E \text{ a } j\text{-dim.} \\ \text{coord. hyperplane}}} \mathrm{Vol}_j(P_E K_\sigma)\mathrm{Vol}_{n-j}(P_{E^\perp} K_{-\sigma}).$$

With our observation about coordinates, we find that for the coordinate sign τ that is defined as $\tau|_E = \sigma|_E$ and $\tau|_{E^\perp} = -\sigma|_{E^\perp}$,

$$\mathrm{Vol}_j(P_E K_\sigma)\mathrm{Vol}_{n-j}(P_{E^\perp} K_{-\sigma}) = \mathrm{Vol}_j(P_E K_\tau)\mathrm{Vol}_{n-j}(P_{E^\perp} K_\tau)$$

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Combining the former with the Rogers-Shephard inequality for sections and projections, $\mathrm{Vol}_j(K \cap K)\mathrm{Vol}_{n-j}(P_{E^\perp} K) \le \binom{n}{j}\mathrm{Vol}(K)$, we continue

$$V_n(K[j], -K[n-j])$$

$$\le \sum_{\sigma \in \{-1,1\}^n} \binom{n}{j}^{-1} \sum_{\substack{E \text{ a } j\text{-dim.} \\ \text{coord. hyperplane}}} \mathrm{Vol}_j(P_E K_\sigma)\mathrm{Vol}_{n-j}(P_{E^\perp} K_{-\sigma}) (\star)$$

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What about the equality cases?

Notice that there were only two inequalities:

(⋆⋆) was an application of the Rogers–Shephard inequality for sections and projections: It is simple to show that there is an equality for all subspaces E of dimension j only if K_τ is a simplex for all $\tau \in \{-1, 1\}^n$.

(⋆) was due to the mixed reverse-Kleitman inequality. If both bodies are simplices, this becomes a computation, for which we needed

$$V_n(K[j], \Delta_n[n-j]) = \frac{1}{n!} \max \left\{ \prod_{i \in I} \alpha_i : |I| = j \right\}.$$

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Mixed volume computation

To compute the mixed volume $V(K[j], \Delta_n[n-j])$ we use

Theorem (Bernstein–Khovanskii–Kouchnirenko (BKK))

Given polynomials $f_1, \ldots, f_n \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, let $P_i = NP(f_i)$ be the Newton polytope of f_i in $\mathbb{R}^n$. Then, for generic choices of the coefficients in the f_i, the number of common solutions (with multiplicity) is exactly $V_n(P_1, ..., P_n)$.

The Newton polytope of f is a convex hull of the set

$$\{(\alpha_1, \ldots, \alpha_n) : x_1^{\alpha_1} \cdots x_n^{\alpha_n} \text{ is a monomial of } f\}$$

Recall that we want to compute the mixed volume of Δ, for which the generic polynomial is

$$f(x) = \sum_{i=1}^{n} c_i x_i + c_0, \quad c_i \in \mathbb{C},$$

with the simplex $K = \mathrm{conv}\{0, \alpha_1 e_1, \dots \alpha_n e_n\}$ (assuming WLOG $\alpha_i \geq \alpha_{i+1}$) the generic polynomial for it is

$$g(x) = \sum_{i=1}^{n} c_i x_i^{\alpha_i} + c_0, \quad c_i \in \mathbb{C}.$$

We want to solve $f_1 = \cdots = f_{n-j} = g_1 = \cdots = g_j = 0$, and since $f_1, \dots f_{n-j}$ are linearly independent, we have a system

$$\begin{cases} 0 = c_0^\ell + x_\ell + \sum_{i=1}^{j} c_i^\ell x_i & j+1 \leq \ell \leq n \\ g_\ell = 0 = c_0^\ell + \sum_{i=1}^{j} c_i^\ell x_i^{\alpha_i} + \sum_{i=j+1}^{n} c_i^\ell x_i^{\alpha_i} & 1 \leq \ell \leq j. \end{cases}$$

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Plug $x_\ell = -c_0^\ell - \sum_{i=1}^{j} c_i^\ell x_i$ and get the equivalent

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As before, $NP(f_\ell) = \Delta_n$, but what is $NP(g_\ell)$?

$$\mathrm{conv}\left(\{\alpha_i e_i\}_{i=1}^{j} \cup \left\{\sum_{k=1}^{j} \beta_k e_k : \sum_{k=1}^{j} \beta_k = \alpha_i,\ \text{for } i \geq j+1,\ \beta_i \geq 0\right\}\right)$$

$$= \mathrm{conv}\left(\{\alpha_i e_i\}_{i=1}^{j} \cup \bigcup_{i=j+1}^{n} \alpha_i \Delta_j\right).$$

$$\subseteq \mathrm{conv}\left(\{\alpha_i e_i\}_{i=1}^{j}\right) = \tilde{K}$$

The inclusion is due to the fact that if $i \geq j$ then by our assumption $\alpha_i \leq \alpha_j$.

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$$\begin{aligned} &\operatorname{conv}\left(\{\alpha_i e_i\}_{i=1}^{j} \cup \left\{\sum_{k=1}^{j} \beta_k e_k : \sum_{k=1}^{j} \beta_k = \alpha_i, \text{ for } i \geq j+1, \ \beta_i \geq 0\right\}\right) \\ &= \operatorname{conv}\left(\{\alpha_i e_i\}_{i=1}^{j} \cup \bigcup_{i=j+1}^{n} \alpha_i \Delta_j\right). \\ &\subseteq \operatorname{conv}\left(\{\alpha_i e_i\}_{i=1}^{j}\right) = \tilde{K} \end{aligned}$$

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We found

$$V_n(\tilde{K}[j], \Delta_n[n-j]) = V_n(K[j], \Delta_n[n-j]).$$

Now using a classical computation, $\tilde(K)$ is j-dimensional, so

$$\begin{aligned} &V_n(\tilde{K}[j], \Delta_n[n-j]) \\ &= \binom{n}{j}^{-1} V_j(\tilde{K}[j]) V_{n-j}(P_{E^\perp}\Delta_n[n-j])) \\ &= \frac{j!(n-j)!}{n!} \mathrm{Vol}_j(\tilde{K}) \mathrm{Vol}_{n-j}(\Delta_{n-j}) = \frac{\prod_{i=1}^{j} \alpha_i}{n!}. \end{aligned}$$

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$$V_n(\tilde{K}[j], \Delta_n[n-j]) = V_n(K[j], \Delta_n[n-j]).$$

Now using a classical computation, $\tilde(K)$ is j-dimensional, so

$$\begin{aligned} &V_n(\tilde{K}[j], \Delta_n[n-j]) \\ &= \binom{n}{j}^{-1} V_j(\tilde{K}[j]) V_{n-j}(P_{E^\perp}\Delta_n[n-j])) \\ &= \frac{j!(n-j)!}{n!} \mathrm{Vol}_j(\tilde{K}) \mathrm{Vol}_{n-j}(\Delta_{n-j}) = \frac{\prod_{i=1}^{j} \alpha_i}{n!}. \end{aligned}$$

Corollary

Let $(\alpha_i)_{i=1}^n, (\beta_i)_{i=1}^n$ be two sequences of non-negative numbers, and let $K, T \subset \mathbb{R}^n$ be given by $K = \mathrm{conv}(0, \alpha_1 e_1, \ldots, \alpha_n e_n)$ and $T = \mathrm{conv}(0, \beta_1 e_1, \ldots, \beta_n e_n)$. Then, for any $0 \leq j \leq n$,

$$V_n(K[j], T[n-j]) = \frac{1}{n!} \max \left\{ \prod_{i \in I} \alpha_i \prod_{j \in I^c} \beta_j : I \subset [n],\ |I| = j \right\}.$$

Thanks for listening

shayas1@gmail.com