Functional volume product, regularizing effect of heat flow, and Brascamp—Lieb inequality

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AGA seminar

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This talk is based on the joint work with Hiroshi Tsuji (Osaka).

Overview

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and exhibit of a wealth of this new link.
- All results are based on a simple observation: for $f_K(x) := e^{-\frac{1}{2}\|x\|_K^2}$,

$$\lim_{s \downarrow 0} c_s \Big(\int_{\mathbb{R}^n} f_K \, dx \Big)^{-\frac{q_s}{p_s}} \Big\| P_s \Big[\Big(\frac{f_K}{\gamma} \Big)^{\frac{1}{p_s}} \Big] \Big\|_{L^{q_s}(\gamma)}^{q_s} = v(K)$$

where $p_s \sim 2s, q_s \sim -2s$ and c_s is explicit. A source of the idea of this identity: Bobkov–Gentil–Ledoux (Hamilton–Jacobi equation).

Inequalities of the volume product

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- (Inverse Santaló inequality, Mahler's conjecture)

$$\inf_{K:K=-K} v(K) =_? v(\mathbb{B}_1^n) = \frac{4^n}{n!}.$$

- The case $n = 2$ was proved by Mahler. After partial progresses by Barthe–Fradelizi, Bourgain–Milman, Fradelizi–Meyer, Kurperberg, Nazarov–Petrov–Ryabogin–Zvavitch,.. the case $n = 3$ was solved by Iriyeh–Shibata '20 and short proof was give by Fradelizi–Hubard–Meyer–Roldán-Pensado–Zvavitch '22.

Functional volume product

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- (Norm of a convex body) For a symmetric convex body K,

$$\|x\|_K := \inf\{r > 0 : \ x \in rK\}, \quad x \in \mathbb{R}^n$$

$$\rightsquigarrow \quad \int_{\mathbb{R}^n} e^{-\frac{1}{2}\|x\|_K^2}\, dx = (2\pi)^{\frac{n}{2}} \frac{|K|}{|\mathbb{B}_2^n|}.$$

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- (Ball and Artstein-Avidan–Klartag–Milman) For $f = e^{-\phi} : \mathbb{R}^n \to \mathbb{R}_+$,

$$v(f) := \int_{\mathbb{R}^n} f\, dx \int_{\mathbb{R}^n} f^\circ\, dx := \int_{\mathbb{R}^n} e^{-\phi}\, dx \int_{\mathbb{R}^n} e^{-\phi^*}\, dx.$$

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- (Passage from functional volume product to geometrical one)

$$v(e^{-\frac{1}{2}\|\cdot\|_K^2}) = c_n v(K), \quad c_n := \frac{(2\pi)^n}{|\mathbb{B}_2^n|^2}.$$

Functional Blaschke–Santaló inequality

Theorem 1 (Ball, Artstein-Avidan–Klartag–Milman, Fradelizi–Meyer, Lehec)

For any even function f,

$$v(f) \leq v(\gamma) = (2\pi)^n$$

and equality iff $f = \gamma_A(x) := (\det 2\pi A)^{-\frac{1}{2}} e^{-\frac{1}{2}\langle x, A^{-1}x\rangle}$ *for some* $A > 0$.
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- Any monotonicity statment of $v(f)$?
- E.g. It is monotone increasing via Steiner symmetrization (Artstein-Avidan–Klartag–Milman) which reduces to the case $n = 1$.

$\rightsquigarrow$ Suggest heat flow monotonicity.

Monotonicity of the functional volume product

For an initial data $f_0 \in L^1(dx)$, let f_t $(t > 0)$ be a Fokker–Planck flow:

$$\partial_t f_t = \mathcal{L}^* f_t := \Delta f_t + \langle x, \nabla f_t \rangle + n f_t.$$

Theorem 2 (N–Tsuji)

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One (technical?) difficulty: $v(e^{-\phi}) = \int e^{-\phi}\, dx \int e^{-\sup_y \langle x,y \rangle - \phi(y)}\, dx$ involves sup $\rightsquigarrow$ doesn't behave well for the integration by parts etc.

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$$\frac{1}{p_s} \to +\infty, \quad p_s \mathcal{Q}_s \to -\frac{1}{2\pi}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{as} \quad s \downarrow 0.$$

E.g. $p_s := 1 - e^{-2s} \sim 2s, \quad \mathcal{Q}_s := \frac{1}{2\pi p_s}\begin{pmatrix} 0 & -e^{-s} \\ -e^{-s} & 0 \end{pmatrix}.$

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$$\Big(\int_{\mathbb{R}^2} e^{-\pi \langle x, \mathcal{Q}_s x \rangle} f_1(x_1)^{\frac{1}{p_s}} f_2(x_2)^{\frac{1}{p_s}} \, dx \Big)^{p_s}$$

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$$f_1 = e^{-\phi^*},\ f_2 = e^{-\phi} \rightsquigarrow \lim_{s\downarrow 0} \Big(\int_{\mathbb{R}^2} e^{-\pi\langle x, \mathcal{Q}_s x\rangle} f_1(x_1)^{\frac{1}{p_s}} f_2(x_2)^{\frac{1}{p_s}}\, dx \Big)^{p_s} = 1.$$

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- Apply Lieb's type theorem (the best cosnt is exhausted by centered Gaussians) and identify BL_s ...? $\to$ A study of the inverse Brascamp–Lieb inequality.
- Prékopa–Leindler = limiting case of the sharp reverse Young (Brascamp–Lieb).

General theory on inverse Brascamp–Lieb: Barthe–Wolff

- **IF** one could have Lieb's type result for this specific BL data:

$$\inf_{f_i \in L^1} \frac{\int_{\mathbb{R}^{2n}} e^{-\pi\langle x, \mathcal{Q}_s x\rangle} f_1(x_1)^{\frac{1}{p_s}} f_2(x_2)^{\frac{1}{p_s}}\, dx}{\prod_{i=1,2} \left(\int_{\mathbb{R}^n} f_i\, dx_i \right)^{\frac{1}{p_s}}} = \inf_{A_1, A_2 > 0} \Lambda_s(\gamma_{A_1}, \gamma_{A_2}),$$

for each $s > 0$, then this would be enough to derive $v(f) \leq v(\gamma)$.

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  for each $s > 0$, then this would be enough to derive $v(f) \leq v(\gamma)$.
- Comprehensive study of the inverse BL ineq by Barthe–Wolff: let $L_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$, $c_1, \ldots, c_m \in \mathbb{R} \setminus \{0\}$, and $\mathcal{Q}$: $n \times n$ symmetric. Then

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- Our specific data fails to satisfy the non-degenerate condition $\rightsquigarrow$ Need to go beyond the condition to enter convex geometry world!

Functional volume product $\leftrightarrow$ regularization of OU flow

$$p_s := 1 - e^{-2s}, \quad \mathcal{Q}_s := \frac{1}{2\pi p_s} \begin{pmatrix} 0 & -e^{-s}\mathrm{id}_{\mathbb{R}^n} \\ -e^{-s}\mathrm{id}_{\mathbb{R}^n} & 0 \end{pmatrix}.$$

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- In fact, we are guided to this specific choice of p_s and $\mathcal{Q}_s$ by a nature of the Orntein–Uhlenbeck flow: for each $s > 0$,

$$P_s g(x) := \int_{\mathbb{R}^n} g(e^{-s}x + \sqrt{1 - e^{-2s}}y)\, d\gamma(y), \quad x \in \mathbb{R}^n,$$

which is a sol to $\partial_s u_s = \Delta u_s - x \cdot \nabla u_s,\ u_0 = g$.

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In fact, our BL is a dual form of the reverse hypercontractivity: for given f_0

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Our inverse BL: $\int e^{-\pi\langle x, \mathcal{Q}_s x\rangle} f_1(x_1)^{\frac{1}{p_s}} f_2(x_2)^{\frac{1}{p_s}}\, dx \ge \mathrm{BL}_s \prod_{i=1,2}\big(\int f_i\big)^{\frac{1}{p_s}}$ is reduced to the $L^p - L^q$ bound of P_s:

$$\big\| P_s\big[\big(\frac{f_0}{\gamma}\big)^{\frac{1}{p_s}}\big]\big\|_{L^{q_s}(\gamma)} \ge \frac{\mathrm{BL}_s}{C_s}\Big(\int_{\mathbb{R}^n} \frac{f_0}{\gamma}\, d\gamma\Big)^{\frac{1}{p_s}}.$$

Borell's reverse hypercontractivity

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- (Borell's reverse hypercontractivity) Suppose $s > 0$ and $q < 0 < p < 1$ satisfy

$$\text{(Nelson's time)} \qquad q \geq q(s,p), \quad q(s,p) := 1 + e^{2s}(p-1).$$

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$\rightsquigarrow$ Rev HC for smaller $q < 0$ quantifies stronger regularization of P_s. Limitation of the regularization is up to $q \geq q(s,p)$.

Improvement of Borell's reverse hypercontractivity

- Our expected rev HC:

$$\big\| P_s \big[\big(\frac{f_0}{\gamma} \big)^{\frac{1}{p_s}} \big] \big\|_{L^{q_s}(\gamma)} \geq \frac{\mathrm{BL}_s}{C_s} \Big(\int_{\mathbb{R}^n} \frac{f_0}{\gamma} \, d\gamma \Big)^{\frac{1}{p_s}}, \quad p_s := 1 - e^{-2s}, \; q_s = 1 - e^{2s}$$

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Theorem 3 (N–Tsuji)

Let $s > 0$ and $1 - e^{2s} \leq q < 0 < p \leq 1 - e^{-2s}$. Then for any even f_0,

$$\Big\| P_s\Big[\Big(\frac{f_0}{\gamma}\Big)^{\frac{1}{p}}\Big]\Big\|_{L^{q}(\gamma)} \geq \Big(\int_{\mathbb{R}^n}\frac{f_0}{\gamma}\,d\gamma\Big)^{\frac{1}{p}}.$$

Moreover, the range of $q < 0 < p$ is best possible. Equality when $f_0 = \gamma$.

Monotonicity statement

- The convex geometrical argument due to Lehec (Prékopa–Leindler + Yao–Yao equipartition) is applicable to the problem of rev HC but the yielding range of p, q is not sharp: $q \geq -p$ and $p \leq 1 - e^{-2s}$.

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Theorem 4 (N–Tsuji)

Let $s > 0$ and $p_s := 1 - e^{-2s}$, $q_s = p_s' = 1 - e^{2s}$. Then for any even f_0,

$$[0, \infty) \ni t \mapsto Q_s(t) := \Big\| P_s\Big[\Big(\frac{f_t}{\gamma}\Big)^{\frac{1}{p_s}}\Big]\Big\|_{L^{q_s}(\gamma)}^{q_s}$$

is monotone increasing where f_t is FP flow: $\partial_t f_t = (\Delta + x \cdot \nabla + n) f_t$.

Monotonicity of the functional volume product (again)

- Recall our observation:

$$\lim_{s \downarrow 0} \Big(\int_{\mathbb{R}^{2n}} e^{-\pi \langle x, \mathcal{Q}_s x \rangle} f_1(x_1)^{\frac{1}{p_s}} f_2(x_2)^{\frac{1}{p_s}} \, dx \Big)^{p_s} = \sup_x f_1(x) f_2^\circ(x).$$

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- Following similar idea, one can show

$$\lim_{s\downarrow 0} c_s\Big(\int_{\mathbb{R}^n} f_0\,dx\Big)^{-\frac{q_s}{p_s}} \Big\| P_s\Big[\Big(\frac{f_0}{\gamma}\Big)^{\frac{1}{p_s}}\Big]\Big\|_{L^{q_s}(\gamma)}^{q_s} = v(f_0) := \int f_0\,dx \int f_0^\circ\,dx$$

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- The monotonicity scheme $Q_s(t) := \Big\| P_s \Big[\Big(\frac{f_t}{\gamma} \Big)^{\frac{1}{p_s}} \Big] \Big\|_{L^{q_s}(\gamma)}^{q_s}$ is introduced by Aoki–Bennett–Bez–Machihara–Matsuura–Shiraki where they proved the monotonicity under Nelson's time condition.

Proof of the monotonicity

Goal: $\frac{d}{dt}\widetilde{Q_s}(t) \geq 0$ where $(p_s, q_s) = (1 - e^{-2s}, 1 - e^{2s})$ and

$$\widetilde{Q_s}(t) := \log\, Q(s) = \log\, \Big\| P_s\Big[\Big(\frac{f_t}{\gamma}\Big)^{\frac{1}{p_s}}\Big]\Big\|_{L^{q_s}(\gamma)}^{q_s},\ \partial_t f_t = (\Delta + x\cdot\nabla + n)f_t.$$

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- Although the rev HC at $(p,q) = (p_s, q_s)$ is the strongest ineq, it appears to be the "easiest" to prove (things becomes canonical).
- A virtue of this specific choice: from $\partial_t f_t = \mathcal{L}^* f_t$,

$$C_s \frac{d}{dt}\widetilde{Q_s}(t) = -\frac{1}{p^2}\int x^2 F_t^q\, dx - \frac{2}{p}\Big(\frac{1}{p} - 1\Big)\int F_t^q \Big(x \int e^{-\frac{1}{p}xz} f_t(z)^{\frac{1}{p}} (\log\, f_t(z$$

Proof of the monotonicity

Goal: $\frac{d}{dt}\widetilde{Q_s}(t) \geq 0$ where $(p_s, q_s) = (1 - e^{-2s}, 1 - e^{2s})$ and

$$\widetilde{Q_s}(t) := \log\ Q(s) = \log\ \Big\| P_s\Big[\Big(\frac{f_t}{\gamma}\Big)^{\frac{1}{p_s}}\Big]\Big\|_{L^{q_s}(\gamma)}^{q_s},\ \partial_t f_t = (\Delta + x\cdot\nabla + n)f_t.$$

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$$F_t(x) := \frac{1}{Z_t}\int_{\mathbb{R}} e^{\frac{1}{p_s}xz} f_t(z)^{\frac{1}{p_s}}\, dz,\quad Z_t := \Big\|\int_{\mathbb{R}} e^{\frac{1}{p_s}xz} f_t(z)^{\frac{1}{p_s}}\, dz\Big\|_{L^{q_s}(dx)}.$$

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- Apply P-BL with $F = F_t^{q_s}$ and $\phi(x) = x$. Notice $F_t^{q_s} = F_t^{q_s}(-\cdot)$ so $\int x\, F_t^{q_s} dx = 0 \rightsquigarrow$

$$\int x^2\, F_t^{q_s} dx \le \int \frac{1}{(-\log\, F_t^{q_s})''}\, F_t^{q_s} dx.$$

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$$\int x^2 F_t^{q_s} dx \leq \int \frac{1}{(-\log F_t^{q_s})''} F_t^{q_s} dx, \quad F_t(x) := \frac{1}{Z_t} \int_{\mathbb{R}} e^{\frac{1}{p_s} xz} f_t(z)^{\frac{1}{p_s}} dz$$

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$$(-\log F_t^{q_s})''(x) = -\frac{q_s}{p_s^2} \Big(\int z^2 G_{x,t}(z) dz - \big(\int z \, G_{x,t}(z) dz \big)^2 \Big),$$

$$G_{x,t}(z) := \frac{1}{\int e^{\frac{1}{p_s} xy} f_t(y)^{\frac{1}{p_s}} \, dy} e^{\frac{1}{p_s} xz} f_t(z)^{\frac{1}{p_s}}.$$

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Apply Cremér–Rao,

$$(-\log\, F_t^{q_s})''(x) = -\frac{q_s}{p_s^2}\mathrm{Var}(G_{x,t}) \ge -\frac{q_s}{p_s^2}\Big(\int \frac{1}{p_s}(-\log\, f_t)''(z)\, G_{x,t}(z)dz\Big)^{-1}$$

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$$\rightsquigarrow \int x^2 F_t^{q_s} dx \le -\frac{p_s}{q_s} \int \big(\int (-\log f_t)''(z) G_{x,t}(z) dz \big) F_t^{q_s} dx$$

Proof of the monotonicity

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Overall,

$$C_s \frac{d}{dt} \widetilde{Q_s}(t) = -\int_{\mathbb{R}} x^2 F_t(x)^{q_s} \, dx$$
$$- (1 - p_s) \int \Big(\int e^{\frac{1}{p_s} xz} f_t(z)^{\frac{1}{p_s}} (\log f_t)(z)'' \, dz \Big) F_t(x)^{q-1} \, dx$$
$$\ge \Big(-\frac{p_s}{q_s} - 1 + p_s\Big) \int \big(\cdots \big) \, dx = 0.$$

Stability of the functional BS

Theorem 5 (Barthe–Böröczky–Fradelizi)

There exists $\varepsilon_0 = \varepsilon_0(n) > 0$ s.t. if ϕ_0 is even convex and satisfies

$$\frac{v(\gamma)}{v(e^{-\phi_0})} < \frac{1}{1-\varepsilon}$$

for some $\varepsilon \in (0, \varepsilon_0)$ then

$$\inf_{B,\mu} \int_{|x| \le R(\varepsilon)} \Big| \frac{1}{2}|x|^2 - \phi_0(Bx) + \mu \Big| \, dx \le C(n) \varepsilon^{\frac{1}{129n^2}}.$$

Here $R(\varepsilon) \le \frac{1}{8n}(\log \frac{1}{\varepsilon})^{\frac{1}{2}}$ and satisfies $\lim_{\varepsilon \to 0} R(\varepsilon) = +\infty$.

Barthe–Böröczky–Fradelizi conjectured that the power of the deficit $\frac{1}{129n^2}$ can be replaced by some absolute constant independent of n. They considered more general functional ineq.

Stability of the functional BS: Wealth of monotonicity

We confirm their conj for uniformly log-concave functs: for $\lambda, \lambda^\circ > 0$,

$$\mathcal{F}(\lambda, \lambda^\circ) := \{\phi : \ \lambda \le \nabla^2 \phi, \ \lambda^\circ \le \nabla^2 \phi^*\}.$$

E.g. Eldan–Mikulincer: dimension free stability for Shannon–Stam.

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Theorem 6 (N–Tsuji)

There exists $\varepsilon_0 = \varepsilon_0(n, \lambda\lambda^\circ)$ *s.t.: If* $\phi_0 \in \mathcal{F}(\lambda, \lambda^\circ)$ *is even and satisfies*

$$\frac{v(\gamma)}{v(e^{-\phi_0})} < e^{\varepsilon} \sim 1 + \varepsilon$$

for some $\varepsilon \in (0, \varepsilon_0)$, *then*

$$\inf_{B,\mu} \int_{|x| \leq R(\varepsilon)} \left| \frac{1}{2}|x|^2 - \phi_0(Bx) + \mu \right| dx \leq C(n, \lambda\lambda^\circ)\varepsilon^{\frac{1}{7}}$$

where $R(\varepsilon) = \frac{\lambda\lambda^\circ}{100}(\log \frac{1}{\varepsilon})^{\frac{1}{2}}$ *and so* $\lim_{\varepsilon \to 0} R(\varepsilon) = +\infty$.

Stability of the functional BS: Main ingredient

Theorem 7 (Cordero-Erausquin)

Let $V \in C^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ be nonnegative, $\int e^{-V}\, dx = 1$ and strictly log-concave. Then for any locally Lipschitz $g \in L^2(hdx)$,

$$\int_{\mathbb{R}^n} |g|^2\, e^{-V} dx - \Big(\int_{\mathbb{R}^n} g\, e^{-V} dx \Big)^2$$
$$\le \int \langle \nabla g, \nabla^2 V^{-1} \nabla g \rangle\, e^{-V} dx - c(h) \int_{\mathbb{R}^n} \big| g(x) - \langle u_0, \nabla V(x) \rangle \big|^2\, e^{-V} dx$$

where

$$u_0 := \int_{\mathbb{R}^n} y g(y)\, e^{-V} dy, \;\; c(h) := \frac{c\lambda(V)}{\sup_x \lambda_{\max}(\nabla^2 V(x)) + c\lambda(V)},$$

c is a numerical constant, $\lambda(V)$ denotes its Poincaré constant, and $\lambda_{\max}(A)$ denotes the maximum eigenvalue of a symmetric matrix A.

Broad picture

- Barthe–Wolff's inverse Brascamp–Lieb inequality (General):

$$(*) \quad \inf_{f_i:\text{arbitrary}} \frac{\int_{\mathbb{R}^n} e^{-\pi\langle x, \mathcal{Q}x\rangle} \prod_{i=1}^m f_i(L_i x)^{c_i}\, dx}{\prod_{i=1}^m \big(\int_{\mathbb{R}^{n_i}} f_i \, dx_i \big)^{c_i}} = \inf_{A_i>0} \Lambda(\gamma_{A_1}, \dots, \gamma_{A_m}),$$

if the data $(\mathbf{c}, \mathbf{L}, \mathcal{Q})$ is non-degenerate in BW sense.

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$$\inf_{f_i:\text{even}} \frac{\int_{\mathbb{R}^n} e^{-\pi\langle x, \mathcal{Q}x\rangle} \prod_{i=1}^m f_i(L_i x)^{c_i}\, dx}{\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i\, dx_i\right)^{c_i}} = \inf_{A_i>0} \Lambda(\gamma_{A_1}, \ldots, \gamma_{A_m})$$

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- (Importance) If one could prove this, one would also solve Kolesnikov–Werner's conjecture about Blaschke–Santaló inequality for multiple convex bodies.

Kolesnikov–Werner's conjecture

Simplest non-trivial case: If f_1, f_2, f_3: even and satisfy

$$\prod_{i=1}^{3} f_i(x_i) \le \exp\Big(-\frac{1}{3-1} \sum_{1\le i<j\le 3} \langle x_i, x_j\rangle \Big), \quad x_1, x_2, x_3 \in \mathbb{R}^n,$$

then

$$\prod_{i=1}^{3} \int_{\mathbb{R}^n} f_i \, dx_i \le \Big(\int_{\mathbb{R}^n} e^{-\frac{1}{2}|x|^2} \, dx \Big)^3 = (2\pi)^{\frac{3n}{2}}.$$

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This would follow from the conjectural inv BL with a data

$$L_i(x_1, x_2, x_3) = x_i, \;\; c_i = \frac{1}{1-e^{-2s}}, \;\; \mathcal{Q}_s = -\frac{e^{-s}}{2\pi(3-1)(1-e^{-2s})} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and then take $s \to 0$.

Thank you for your attention.