# Symmetrization Resistance

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# Symmetrization resistance

**Definition**

$Z \in_R \mathbb{R}$ is *symmetric about zero* iff $\forall z \quad \mathbb{P}(Z \leq -z) = \mathbb{P}(Z \geq z)$

- For discrete $Z \in_R \mathbb{R}$, denote its PMF by $f_Z$
- Equivalent for discrete $Z$:

**Definition**

$\forall z \in \operatorname{supp}(f_Z), \quad f_Z(-z) = f_Z(z)$ (*symmetry equations*)

**Definition**

For discrete $X \in_R \mathbb{R}$, a *symmetrizer* is an independent $Y \in_R \mathbb{R}$ such that $X + Y$ is symmetric about zero.

## Definition

Discrete $X \in_R \mathbb{R}$ is...

- *variance symmetrization resistant* iff all symmetrizers $Y$ satisfy $\text{Var}(Y) \geq \text{Var}(X)$
- *entropic symmetrization resistant* iff all symmetrizers $Y$ satisfy $H(Y) \geq H(X)$

Continuous question is also interesting.

Motivation:

- Original question of KMSVV99: Gaussianization. Given non-Gaussian $X$, how can we choose $Y$ such that $X + Y$ is "as Gaussian as possible"?
- If we use KL-divergence from Gaussian, equivalent to problem of maximizing capacity of additive noise channel with noise $X$
  - $X$ is noise, $Y$ is signal
  - Transmit power constrains $Y$
- Recent work on Gaussian mixtures and additive noise: Eskenazis, Nayar & Tkocz 2018; Madiman, Nayar & Tkocz 2019, 2021

# Symmetrization Resistance: Known Results

- The only distributions on $\mathbb{R}$ known to be symmetrization resistant are Bernoulli.
- Notation: $X \sim \text{Bernoulli}(p, a, b)$ $(a < b)$:

$$\mathbb{P}(X = a) = q \text{ and } \mathbb{P}(X = b) = p.$$

- Notation: $q = 1 - p$.

**Theorem (Kagan, Mallows, Shepp, Vanderbei and Vardi 1999)**

*Asymmetric Bernoulli r.v.s are variance symmetrization resistant.*

- Proof: exhibited solution to linear program.
- Second proof: Pal 2008 (stochastic calculus; Skorokhod embedding).
- Third proof: Madiman and Pollard 2023 (find basis for affine hull of space of symmetrizers; bound coefficients)

**Theorem (Madiman and Pollard 2023)**

*Asymmetric Bernoulli r.v.s are entropic symmetrization resistant.*

In both cases, equality iff $f_Y = f_{-X}$.

# Known negative results

- Symmetric integrable $X \in_R \mathbb{R}$ are never symmetrization resistant.
  - $-\mathbb{E}X$ is a symmetrizer
  - $\mathrm{Var}(-\mathbb{E}X) = H(-\mathbb{E}X) = 0$
- Definition: $X$ has a *symmetric component* if there exist independent $U$ and $V$ (symmetric $V$) and $X = U + V$.

**Lemma (Kagan, Mallows, Shepp, Vanderbei and Vardi 1999)**

*$X$ has nontrivial symmetric component $\Rightarrow$ not variance symm. res.*

**Lemma**

*$X$ has nontrivial symmetric component $\Rightarrow$ not entropic symm. res.*

- Note: For Bernoulli, $f_X$ has nontrivial symmetric component iff $f_X$ is symmetric.

# Known negative results: Binomial

- KMSVV (1999) showed asymmetric $f_X \sim \text{Binomial}(n, p)$ with $n \geq 4$ and and $p \in (0.489, 0.5)$ are not variance symm. res.
- We believe these are not entropic symm. res. either (numerical support)
- Asymmetric Binomial with $n = 2, 3$ open.

# Elementary observations

- Notation: convolution
  for PMFs $f, g$ on $\mathbb{R}$,

$$(f * g)(u) = \sum_{w \in \mathrm{supp}(g)} f(u-w)g(w) = \sum_{w \in \mathrm{supp}(f)} f(w)g(u-w)$$

- $Y \sim f$ symmetrizes $X$
  iff $f$ symmetrizes $f_X$
  iff $f * f_X$ is symmetric about zero
- $|\mathrm{supp}(X)| = 2$: sufficient to investigate $X \sim \mathrm{Bernoulli}(p, -1, 1)$; $p > \frac{1}{2}$.

# The space $\mathcal{Y}$ of symmetrizer PMFs

- Notation: for $X \in_R \mathbb{R}$,

$$\mathcal{Y} = \mathcal{Y}[f_X] = \{\text{PMFs } f \mid f * f_X \text{ is symmetric about zero }\}.$$

- $\mathcal{Y}$ is convex
- $H$ and Var are concave
- Idea: Krein-Milman?
  - Difficulty 1: Unclear whether $\mathcal{Y}$ is compact
  - Difficulty 2: Many extreme points, some not obvious.
- Solution: find basis in $\mathcal{Y}$ for aff($\mathcal{Y}$); control negative coeffs

# The functions $\hat{f}$

Now let $X \sim \text{Bernoulli}(p, -1, 1)$, $p > \frac{1}{2}$.

Notation:

- Indicator function of $E \subseteq \mathbb{R}$: $\chi_E$
- Point indicator: $\chi_w = \chi_{\{w\}}$ for $w \in \mathbb{R}$

Useful symmetrizer PMFs: For any PMF $f_X$ on $\mathbb{R}$ and any $z \in \mathbb{R}$, define

- $\hat{f}_z(u) = \frac{1}{2}\Big((\chi_{-z} * f_{-X})(u) + (\chi_z * f_{-X})(u)\Big)$

**Lemma (The $\hat{f}$ are symmetrizers)**

*For any PMF $f_X$ on $\mathbb{R}$, for any $z \in \mathbb{R}$, $\hat{f}_z \in \mathcal{Y}[f_X]$.*

Define for $r \in [0,1]$,

- $S^r = 2\mathbb{Z} + \{\pm r\}$ (partition of $\mathbb{R}$)
- $\mathcal{Y}^r = \{f \in \mathcal{Y} \mid \operatorname{supp}(f) \subseteq S^r\}$.
- $I^r = \begin{cases} \{1,2,\dots\} & \text{when } r = 0 \\ \mathbb{Z} & \text{when } r \in (0,1) \\ \{0,1,2,\dots\} & \text{when } r = 1. \end{cases}$
- $\hat{f}^r_k(z) = \hat{f}_{2k+1+r} \in \mathcal{Y}^r$ for all $k \in I^r$
- and $R_f = \{r \in [0,1] \mid \operatorname{supp}(f) \cap S^r \neq \emptyset\}$

**Lemma (Extreme symmetrizer spaces $\mathcal{Y}^r$)**

*$f \in \mathcal{Y}, r \in R_f \Rightarrow$ the unique PMF $f^r \propto f\big|_{S^r}$ satisfies $f^r \in \mathcal{Y}^r$.*
*Also, if $f \notin \mathcal{Y}^r$, then the unique PMF $g \propto f\big|_{\mathbb{R}\setminus S^r}$ satisfies $g \in \mathcal{Y}$, and $\exists c^r \in (0,1)$ s.t. $f = c^r f^r + (1-c^r)g$.*

Proof: symmetry equations respect partition $S^r$.

## Theorem (Representation theorem for $\mathcal{Y}$(Bernoulli))

*For* $f_X \sim \text{Bernoulli}(p, -1, 1)$ *and* $f \in \mathcal{Y}[f_X]$,

$$f = \sum_{r \in R_f} \sum_{k \in I^r} \alpha_k^r \hat{f}_k^r .$$

*Moreover,* $\sum_{r \in R_f} \sum_{k \in I^r} \alpha_k^r = 1$.

Specifically:

- $\alpha_k^r = \frac{2}{p-q}(pf(-2k-r) - qf(2k+r))$,
- except $\alpha_1^0 = \frac{1}{p-q}(pf(-2k-r) - qf(2k+r))$.

Proof:

- First prove for $f \in \mathcal{Y}^r$, then sum over $R_f$
- For $f \in \mathcal{Y}^r$: prove for finite dimensional spaces $\mathcal{Y}_n^r = \{f \in \mathcal{Y}^r \mid \operatorname{supp}(f) \subseteq [-2n-r, 2n+r]\}$
- Then take $n \to \infty$.

## Lemma (Negative coefficient control)

*Let*

$$f = \sum_{r \in R_f} \sum_{k \in I^r} \alpha_k^r \hat{f}_k^r \in \mathcal{Y}.$$

*If* $\alpha_j^r \leq 0$*:*

- $\alpha_{j+1}^r \geq \frac{p}{q}\left|\alpha_j^r\right| > \left|\alpha_j^r\right|$
- *also* $\alpha_{j-1}^r \geq \frac{p}{q}\left|\alpha_j^r\right| > \left|\alpha_j^r\right|$ *when it exists (i.e. when* $j-1 \in I^r$*)*

*Also,* $\alpha_1^0 \geq 0$.

## Proof.

From symmetry equations. □

## Theorem (Entropic symm. res. of Bernoulli)

*Asymmetric* **Bernoulli r.v.s are entropic symmetrization resistant.**

*That is, for* $X \sim \mathrm{Bernoulli}(p, a, b)$ *with* $p \neq \frac{1}{2}$*, any* $f \in \mathcal{Y}[f_X]$ *satisfies* $H(f) \geq H(f_X)$*.*

## Proof (outline).

- Sufficient to investigate $f_X \sim \mathrm{Bernoulli}(p, -1, 1)$, $p > \frac{1}{2}$.
- Any $f \in \mathcal{Y}$ with $H(f) < H(f_X)$ must satisfy $f(0) > 0$
- Therefore sufficient to investigate $\mathcal{Y}^0$ (concavity of entropy)
- Show that $f(0) \geq p > \frac{1}{2}$ for $f \in \mathcal{Y}^0$
- This implies $\frac{1}{2} < f(0) = \alpha_0^1 \hat{f}_0^1(0) = \frac{\alpha_0^1}{2}$, thus $\alpha_0^1 > 1$
- But $\sum_{r,k} \alpha_k^r = 1$, so

$$1 < \alpha_0^1 \leq \alpha_0^1 + \sum_{(r,k) \neq (1,0)} \alpha_k^r = \sum_{r,k} \alpha_k^r = 1, \quad \text{contradiction.} \quad \square$$

## Theorem (Variance symm. res. of Bernoulli)(KMSVV 1999)

*Asymmetric* **Bernoulli r.v.s are variance symmetrization resistant.**

*That is, for* $X \sim \text{Bernoulli}(p, a, b)$ *with* $p \neq \frac{1}{2}$*, any* $f \in \mathcal{Y}[f_X]$ *satisfies* $\text{Var}(f) \geq \text{Var}(f_X)$.

## New proof (1/2)

- Sufficient to investigate $X \sim \text{Bernoulli}(p, -1, 1)$, $p > \frac{1}{2}$
- Sufficient to investigate second moment $M_2$
- Concavity of variance: sufficient to investigate $f \in \mathcal{Y}^0$

## New proof (2/2).

- For $f \in \mathcal{Y}^0$, compute

$$
\begin{aligned}
M_2(f) = \sum_{z \in I^0 = 2\mathbb{Z}} z^2 f(z) &\geq 4 \sum_{z \neq 0} f(z) \\
&= 4\Big(1 - f(0)\Big) \\
&= 4\Big(1 - \alpha_1^0 \hat{f}_1^0(0)\Big) \\
&\geq 4\Big(1 - \frac{1}{2}\Big) = 2 = M_2(f_{-X}).
\end{aligned}
$$

□

## Corollary (hypercube, entropy version)

*Let:*

- $X = (X_1, \ldots, X_d) \in_R \{-1, 1\}^d$, *all* $X_i$ *asymmetric*
- $Y = (Y_1, \ldots, Y_d) \in \mathcal{Y}[f_X]$

*Then:*

$$H(Y) \geq \frac{1}{d} H(X).$$

- Constant $\frac{1}{d}$ results from dependence between coordinates
- Rotate, translate, scale

- Define: matrix norm $\|A\|_{1,1} = \sum_{i,j} |A_{ij}|$

**Corollary (Hypercube, variance version)**

*Let:*

- $X = (X_1, \ldots, X_d) \in_R \{-1, 1\}^d$, *all* $X_i$ *asymmetric*
- $Y = (Y_1, \ldots, Y_d) \in \mathcal{Y}[f_X]$

*Then:*

$$\|\mathrm{Cov}(Y)\|_{1,1} \geq \frac{1}{d} \|\mathrm{Cov}(X)\|_{1,1}.$$

- Same constant $\frac{1}{d}$

# Support in arithmetic progression, cardinality 3

- Symm. res. of discrete $X \in_R \mathbb{R}$ with $|\mathrm{supp}(f_X)| = 2$ is solved.
- How to generalize to $X \in_R \mathbb{R}$ with $|\mathrm{supp}(f_X)| = 3$?
- New difficulties:
  - $\mathrm{supp}(f_X)$ might not be an arithmetic progression
    e.g. $\mathrm{supp}(f_X) = \{0, 1, 3\}$
  - $\mathrm{supp}(f_X)$ might not even be *contained* in an arithmetic progression
    e.g. $\mathrm{supp}(f_X) = \{0, 1, \pi\}$
  - possible nontrivial symmetrizers $f \in \mathcal{Y}[f_X]$ with $\mathrm{supp}(f) < \mathrm{supp}(f_X)$
  - symmetric part of asymmetric $f_X$ may now be nontrivial
- New assumptions:
  - Assume $f_X$ has no nontrivial symmetric part
  - Assume $\mathrm{supp}(f_X)$ is an arithmetic progression
  - Equivalently: $\mathrm{supp}(f_X) = \{0, \pm 2\}$
- Other directions possible (Binomial $f_X$, monotone $f_X$, ...)

Redefine:

- $I^r = \begin{cases} \{0, 1, 2, \dots\} & \text{when } r = 0 \text{ or } r = 1 \\ \mathbb{Z} & \text{when } r \in (0, 1). \end{cases}$

**Theorem (Representation theorem)**

*Let $\operatorname{supp}(f_X) = \{-2, 0, 2\}$ and let $f_X$ have no nontrivial symmetric component. Then, for all $f \in \mathcal{Y}[f_X]$,*

$$f = \sum_{r \in R_f} \sum_{i \in I^r} \alpha_i^r \hat{f}_i^r$$

*everywhere on $\mathbb{R}$. Moreover, the coefficients $\alpha_i^r$ are the unique coefficients with this property.*

- Difficulty/complexity seems to increase with $|\operatorname{supp}(f_X)|$.

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# Summary

## Theorems

For asymmetric Bernoulli $X \in_R \mathbb{R}$ and independent $Y \in_R \mathbb{R}$ such that $X + Y$ is symmetric about zero,

- $\mathrm{Var}(Y) \geq \mathrm{Var}(X)$ (KMSVV99)
- $H(Y) \geq H(X)$ (Madiman & Pollard)

with equality iff $f_Y = f_{-X}$.

## Corollaries

For $X \in_R \{-1, 1\}^d$ with asymmetric coordinates and independent $Y \in_R \mathbb{R}^d$ such that $X + Y$ is symmetric about zero,

- $\mathrm{Var}(Y) \geq \frac{1}{d} \mathrm{Var}(X)$
- $H(Y) \geq \frac{1}{d} H(X)$

Thank you!