On the monotonicity of discrete entropy for log-concave random variables on $\mathbb{Z}^d$

Martin Rapaport

Based on joint work with Matthieu Fradelizi and Lampros Gavalakis

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Outline

Differential and Shannon entropies

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The Shannon entropy $\rightarrow$ *uncertainty* or "*surprise*" of a random variable X.

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If X_1, X_2 are identically distributed (1) can be rewritten as

$$h(X_1 + X_2) \geq h(X_1) + \frac{d}{2}\log 2.$$

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Question adressed in this talk: can we extend these results to $\mathbb{Z}^d$?

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This definition coincides with the usual log-concavity in one dimension.

From a Theorem of approximation of entropies to our final result

Theorem (Monotonicity of discrete entropy for log-concave random variables on $\mathbb{Z}^d$)

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Proof.

By the scaling property of the generalisation of continuous EPI one has

$$h\Big(\sum_{i=1}^{n+1} X_i + U_i\Big) \geq h\Big(\sum_{i=1}^{n} X_i + U_i\Big) + \frac{d}{2}\log\left(\frac{n+1}{n}\right)$$

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Then, it suffices to prove the Theorem of approximation of entropies.

A discrete analogue upper bound on the isotropic constant for log-concave functions

Notion of isotropicity and some definitions in the continuous setting

The upper bound will be an essential step for the proof of the approximation Theorem and may be of independent interest.

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In the discrete setting $p : \mathbb{Z}^d \to \mathbb{R}_+$, we define the covariance matrix $\mathrm{Cov}(p)$ by

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A family $\{f_\sigma\}_{\sigma\in\mathbb{R}_+}$ of non-negative functions on $\mathbb{R}^d$ is almost isotropic

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$$\begin{aligned} \mathrm{Cov}(f_\sigma)_{i,j} &= \sigma^2 + O(\sigma) \quad \text{for } i = j, \\ &= O(\sigma) \quad \text{for } i \neq j. \end{aligned}$$

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Theorem

Suppose p *is a log-concave p.m.f. on* $\mathbb{Z}^d$ *with almost isotropic extension and covariance matrix* $\mathrm{Cov}(p)$.

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A small détour: the slicing conjecture

Slicing conjecture: there exists a universal constant $c > 0$ such that, for any dimension d and for any convex body K in isotropic position in $\mathbb{R}^d$ and any direction $\theta \in \mathbb{S}^{d-1}$, one has $|K \cap \theta^{\perp}|_{d-1} \geq c$.

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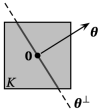

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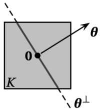

Slicing conjecture for log-concave measures : there exists a universal constant C such that for any dimension d and for any log-concave integrable function $f : \mathbb{R}^d \to [0, \infty)$ one has $L_f \leq C$.

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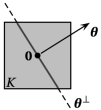

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Best known constant by a recent result of Klartag:

$$L_f \leq C\sqrt{\log d}.$$

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Approximations for the integral, mean and covariance discretely

Let us prove a discrete analogue of the upper bound on the isotropic constant for log-concave functions in the simplest case where f is isotropic log-concave .

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Let us prove a discrete analogue of the upper bound on the isotropic constant for log-concave functions in the simplest case where f is isotropic log-concave . To do so, we must be able to approximate the integral, mean and covariance discretely. We obtain the following results; for this, we will need to use the *Keith Ball's bodies* as well as concentration results.

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$$\left|\int_{\mathbb{R}^d} f - \sum_{\mathbb{Z}^d} f\right| = o_d(1), \quad \text{as } \sigma \to \infty,$$

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$$\left|\det\Big(\mathrm{Cov}_{\mathbb{Z}^d}(f)\Big) - \det\Big(\mathrm{Cov}_{\mathbb{R}^d}(f)\Big)\right| = O_d(\sigma^{2d-2}), \quad \text{as } \sigma \to \infty.$$

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Just for a moment, let's admit these approximations ...

A discrete analogue upper bound on the isotropic constant for log-concave functions

Case of isotropic log-concave functions

Proof.

Since p is extensible log-concave, there exists a continuous log-concave function f (not necessarily a density) such that $f(k) = p(k)$ for all $k \in \mathbb{Z}^d$ and

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$$\max_{k\in\mathbb{Z}^d} p(k) \leq \frac{C_d(1 + O_d(\frac{1}{\sigma}))}{\det\Big(\mathrm{Cov}(p)\Big)^{\frac{1}{2}} + O_d(\sigma^{d-1})} \leq \frac{4C_d}{\det\Big(\mathrm{Cov}(p)\Big)^{\frac{1}{2}}}$$

provided that σ is large enough depending on d,

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provided that σ is large enough depending on d, using that $O_d(\sigma^{d-1}) \geq -\frac{1}{2}\det\Big(\mathrm{Cov}(p)\Big)^{\frac{1}{2}} \simeq -\frac{1}{2}\sigma^d$. ∎

Some tools to approximate the integral, mean and covariance discretely

A lemma using *Keith Ball's bodies*

Assume that $f : \mathbb{R}^d \to \mathbb{R}$ is a centered, isotropic, log-concave density: $\int_{\mathbb{R}^d} f = 1, \int_{\mathbb{R}^d} xf = 0$ and $\int_{\mathbb{R}^d} x^T xf = \sigma^2 \mathrm{I}_d$.

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$f \to$ *Keith Ball's bodies* .

Definition (Keith Ball's bodies)

For any $p > 0$, the set $K_p(f)$ is defined as follows

$$K_p(f) := \left\{ x \in \mathbb{R}^d : \int_0^\infty p r^{p-1} f(rx) dr \geq f(0) \right\}.$$

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Assume that $f : \mathbb{R}^d \to \mathbb{R}$ is a centered, isotropic, log-concave density: $\int_{\mathbb{R}^d} f = 1, \int_{\mathbb{R}^d} xf = 0$ and $\int_{\mathbb{R}^d} x^T xf = \sigma^2 \mathrm{I}_d$.

$f \to$ *Keith Ball's bodies* .

Definition (Keith Ball's bodies)

For any $p > 0$, the set $K_p(f)$ is defined as follows

$$K_p(f) := \left\{ x \in \mathbb{R}^d : \int_0^\infty p r^{p-1} f(rx) dr \geq f(0) \right\}.$$

This important family of bodies was introduced by Keith Ball, who established that the set $K_p(f)$ is a convex body.

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This important family of bodies was introduced by Keith Ball, who established that the set $K_p(f)$ is a convex body. Moreover, its radial function is $\rho_{K_p(f)}(x) = \left(\frac{1}{f(0)} \int_0^\infty p r^{p-1} f(rx) dr \right)^{\frac{1}{p}}$ for $x \neq 0$.

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Lemma

Let $d \geq 1$ *be an integer. There exist two constants* $0 < C'_d < C_d$ *such that for any* $f : \mathbb{R}^d \to \mathbb{R}_+$ *centered, isotropic, log-concave density and for every* $\theta \in \mathbb{S}^{d-1}$,

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$$(C'_d)^d \leq \int_0^{\infty} dr^{d-1} f(r\theta) dr \leq (C_d)^d,$$

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where C_d and C'_d are constants depending only on the dimension d. In fact, we may take

$$C'_d = \frac{c_1^{d+2}}{\sqrt{2\pi} e^{\frac{3}{2}}} \quad \text{and} \quad C_d = (d+1) c_2^{d+2} \max_f L_f,$$

where c_1 and c_2 will be explicit later.

Proof of lemma

The function f being isotropic, we have $\mathrm{Cov}(f) = \sigma^2 I_d$, for some $\sigma > 0$ and $\int f = 1$, thus $L_f = \max(f)^{\frac{1}{d}} \sigma$.

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Theorem (Kannan-Lovász-Simonovits)

Let K be a centered convex body in $\mathbb{R}^d$ and $u \in \mathbb{S}^{d-1}$. Then

$$\frac{h_K(u)^2}{d(d+2)} \leq \frac{1}{|K|} \int_K \langle x, u \rangle^2 dx \leq \frac{d}{d+2} h_K(u)^2.$$

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Using inclusion relations between $K_p(f)$ and $K_q(f)$ and classical inequalities on Gamma functions : there exist $0 < c_1 < c_2$ such that for any dimension $d \geq 1$, one has

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$$\sqrt{\frac{d+2}{d}}c_1^{d+2}\sigma B_2^d \subset K_d(f) \subset \sqrt{d(d+2)}c_2^{d+2}\sigma B_2^d.$$

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$$\frac{L_f}{e} \leq f(0)^{\frac{1}{d}}\sigma = \left(\frac{f(0)}{\max(f)}\right)^{\frac{1}{d}} L_f \leq L_f,$$

since $L_f \geq L_{\mathbb{1}_{B_2^d}} \geq 1/\sqrt{2\pi e}$, we conclude.

Some tools needed for the approximations of the integral, mean and covariance discretely

A concentration lemma

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As a consequence of the previous lemma, we obtain the following concentration lemma.

Lemma (Concentration Lemma)

Let $c_d := 3^{\frac{1}{d}} C_d$. *Then, for every log-concave, isotropic, centered density function* f *and for every* $x \in \mathbb{R}^d$ *such that* $\|x\|_2 \geq c_d / f(0)^{\frac{1}{d}}$,

$$f(x) \leq f(0) 2^{-\|x\|_2 \frac{f(0)^{\frac{1}{d}}}{c_d}}.$$

With all these lemmas, the following approximations can be proved:

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$$\left|\int_{\mathbb{R}^d} f - \sum_{\mathbb{Z}^d} f\right| = o_d(1), \quad \text{as } \sigma \to \infty,$$

$$\left|\int_{\mathbb{R}^d} xf - \sum_{k\in\mathbb{Z}^d} kf(k)\right| = O_d(1), \quad \text{as } \sigma \to \infty,$$

$$\left|\det\left(\mathrm{Cov}_{\mathbb{Z}^d}(f)\right) - \det\left(\mathrm{Cov}_{\mathbb{R}^d}(f)\right)\right| = O_d(\sigma^{2d-2}), \quad \text{as } \sigma \to \infty.$$

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All these arguments can be generalised to almost isotropic log-concave distributions.

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Let $F(x) = x \log \frac{1}{x}, x > 0$.

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$$h(X_1 + \cdots + X_n + U_1 + \cdots + U_n)$$
$$= \sum_{k:\|k\|_2 \le \sigma^2} \int_{k+[0,1)^d} F(f_{S_n+U^n}(x))dx + \sum_{k:\|k\|_2 > \sigma^2} \int_{k+[0,1)^d} F(f_{S_n+U^n}(x))dx.$$

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- Second term: Entropy tails $\to 0$.
- First term $\approx H(S_n)$, for this we will use the discrete analogue upper bound on the isotropic constant for log-concave functions.

An open problem/ Work in progress

For $d = 1$, our definition of log-concavity is equivalent to the usual definition $p(k)^2 \geq p(k-1)p(k+1), k \in \mathbb{Z}$, which is preserved under convolution .

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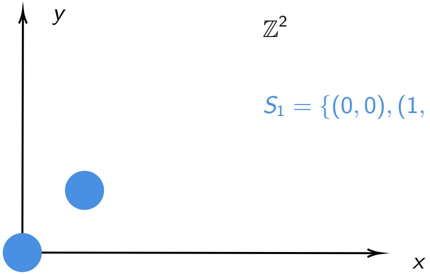

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For $d > 1$ log-concavity may not be preserved in general, considering two log-concave distributions supported on S_1 and S_2 respectively.

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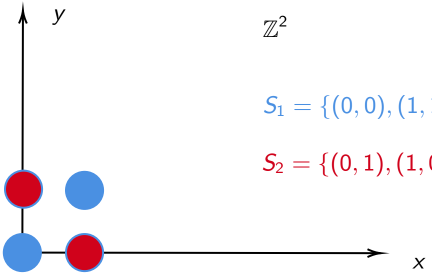

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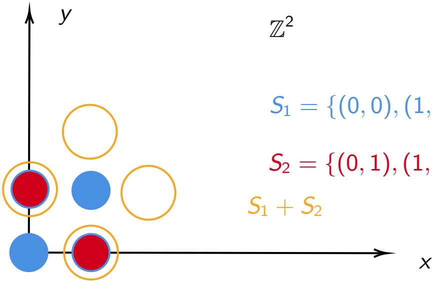

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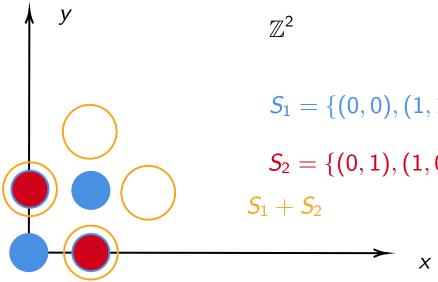
An open problem/ Work in progress

For $d = 1$, our definition of log-concavity is equivalent to the usual definition $p(k)^2 \geq p(k-1)p(k+1), k \in \mathbb{Z}$, which is preserved under convolution . A related concept to the one of extensible log-concave: a set $A \subset \mathbb{Z}^d$ is said to be hole free if $A = \mathrm{conv}(A) \cap \mathbb{Z}^d$.

Minkowski sum of hole free sets can have a hole (example of Murota)!



$\mathbb{Z}^2$

$S_1 = \{(0,0),(1,1)\}$

$S_2 = \{(0,1),(1,0)\}$

$S_1 + S_2$

For $d > 1$ log-concavity may not be preserved in general, considering two log-concave distributions supported on S_1 and S_2 respectively.

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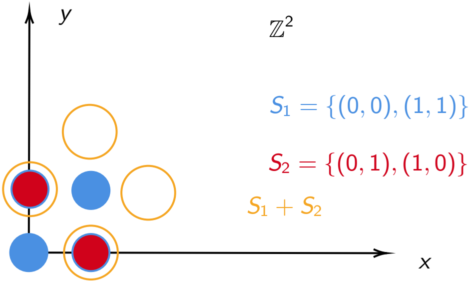

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Our definition of log-concavity is preserved under self-convolution?

Thanks a lot !!!!!!!

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