Asymptotic bounds for the combinatorial diameter of random polytopes

Sophie Huiberts (CWI)

joint work with Gilles Bonnet, Daniel Dadush, Uri Grupel, Galyna Livshyts

TU Berlin, September 14th 2021

A question about optimization

Given a basic feasible solution to

$$\begin{aligned} \text{maximize } & c^{\mathsf{T}} x \\ \text{subject to } & Ax \leq b \end{aligned}$$

how many pivot steps does the simplex method need to find an optimal solution?

- n variables
- m constraints

A question about geometry

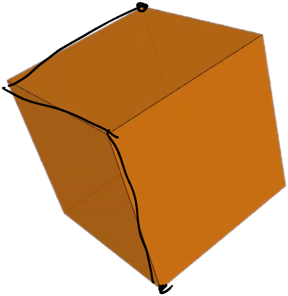
Given a basic feasible solution to

$$\text{maximize } c^\mathsf{T} x$$
$$\text{subject to } Ax \leq b$$

how many pivot steps does the simplex method need to find an optimal solution?

- n variables
- m constraints

Given a polyhedron, how many edges do we need to traverse to go from any vertex to any other?



"Combinatorial diameter"

The Hirsch conjecture

If $P \subset \mathbb{R}^n$ has m facets, is the diameter at most $m - n$?

The Hirsch conjecture

If $P \subset \mathbb{R}^n$ has m facets, is the diameter at most $m - n$?

True for

- polytopes in $\mathbb{R}^3$
- network flow polytopes
- fractional stable set polytopes
- polytopes with vertices in $\{0, 1\}^n$
- and many more

The Hirsch conjecture

If $P \subset \mathbb{R}^n$ has m facets, is the diameter at most $m - n$?

True for

- polytopes in $\mathbb{R}^3$
- network flow polytopes
- fractional stable set polytopes
- polytopes with vertices in $\{0, 1\}^n$
- and many more

But false in general.
Klee-Walkup, 1967, counterexample for unbounded polyhedra.
Santos, 2012, counterexample for bounded polytopes.

The Hirsch conjecture

If $P \subset \mathbb{R}^n$ has m facets, is the diameter at most $m - n$?

True for

- polytopes in $\mathbb{R}^3$
- network flow polytopes
- fractional stable set polytopes
- polytopes with vertices in $\{0, 1\}^n$
- and many more

But false in general.
Klee-Walkup, 1967, counterexample for unbounded polyhedra.
Santos, 2012, counterexample for bounded polytopes.

New conjecture: is the diameter at most $\mathrm{poly}(m, n)$?

Best available diameter bounds

Barnette, Larman:

$$\text{Diameter}(P) \leq 2^{n-2} m.$$

Kalai-Kleitman, Todd, Sukegawa:

$$\text{Diameter}(P) \leq (m-n)^{\log_2 O(n/\log n)}.$$

Best available diameter bounds

Barnette, Larman:

$$\text{Diameter(P)} \leq 2^{n-2}m.$$

Kalai-Kleitman, Todd, Sukegawa:

$$\text{Diameter(P)} \leq (m-n)^{\log_2 O(n/\log n)}.$$

Dyer-Frieze, Bonifas-Di Summa-Eisenbrand-Hähnle-Niemeier, Dadush-Hähnle, Narayanan-Shah-Srivastava:
If A is integral and every absolute square subdeterminant is at most Δ then

$$\text{Diameter(P)} \leq O(n^3\Delta^2\log(\Delta)).$$

Best available diameter bounds

Barnette, Larman:

$$\text{Diameter(P)} \leq 2^{n-2} m.$$

Kalai-Kleitman, Todd, Sukegawa:

$$\text{Diameter(P)} \leq (m-n)^{\log_2 O(n/\log n)}.$$

Dyer-Frieze, Bonifas-Di Summa-Eisenbrand-Hähnle-Niemeier, Dadush-Hähnle, Narayanan-Shah-Srivastava:
If A is integral and every absolute square subdeterminant is at most Δ then

$$\text{Diameter(P)} \leq O(n^3 \Delta^2 \log(\Delta)).$$

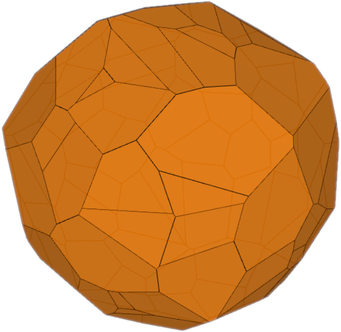
Random polytopes are "well-conditioned on average". Do they have bounded diameter?

Tantalizing evidence from the simplex method

Consider a fixed two-dimensional plane $W \subset \mathbb{R}^n$. Let

$$P = \{x \in \mathbb{R}^n : \langle a, x \rangle \leq 1 \ \forall a \in A\}$$

where $A \subset \mathbb{R}^n$, $|A| = m$ is sampled iid from a rotationally invariant probability distribution.

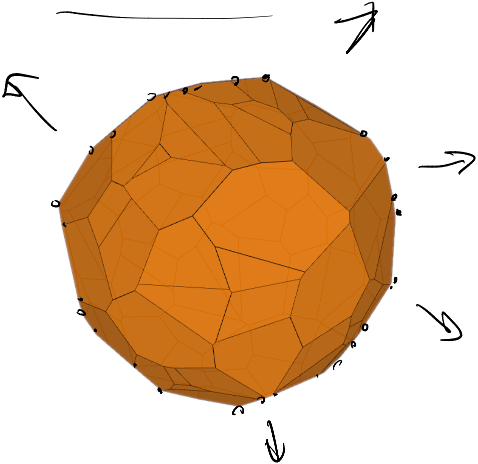


Tantalizing evidence from the simplex method

Consider a fixed two-dimensional plane $W \subset \mathbb{R}^n$. Let

$$P = \{x \in \mathbb{R}^n : \langle a, x \rangle \leq 1 \; \forall a \in A\}$$

where $A \subset \mathbb{R}^n$, $|A| = m$ is sampled iid from a rotationally invariant probability distribution.



Borgwardt 1977,1982,1987,1999: on expectation, $O(n^2 m^{\frac{1}{n-1}})$ distinct vertices are optimized by the objectives in W.

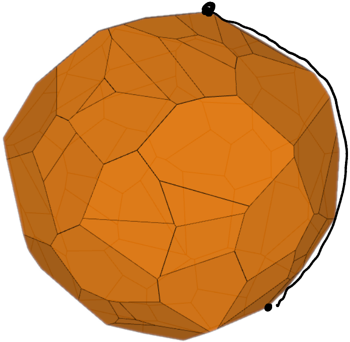
Tantalizing evidence from the simplex method

Consider a fixed two-dimensional plane $W \subset \mathbb{R}^n$. Let

$$P = \{x \in \mathbb{R}^n : \langle a, x \rangle \leq 1 \ \forall a \in A\}$$

where $A \subset \mathbb{R}^n$, $|A| = m$ is sampled iid from a rotationally invariant probability distribution.

$A \approx N(0, I)$,



Borgwardt 1977,1982,1987,1999: on expectation, $\boxed{O(n^2 m^{\frac{1}{n-1}})}$ distinct vertices are optimized by the objectives in W.

The simplex method needs $O(n^2 m^{\frac{1}{n-1}})$ pivot steps in expectation to move from any one such vertex to any other.

Tantalizing evidence from the simplex method

Consider a fixed two-dimensional plane $W \subset \mathbb{R}^n$. Let

$$P = \{x \in \mathbb{R}^n : \langle a, x \rangle \leq 1 \ \forall a \in A\}$$

where $A \subset \mathbb{R}^n, |A| = m$ is sampled independently from Gaussian distributions with covariance $\sigma^2 I_{n \times n}$ and expectation having norm at most 1.

Tantalizing evidence from the simplex method

Consider a fixed two-dimensional plane $W \subset \mathbb{R}^n$. Let

$$P = \{x \in \mathbb{R}^n : \langle a, x \rangle \leq 1 \ \forall a \in A\}$$

where $A \subset \mathbb{R}^n, |A| = m$ is sampled independently from Gaussian distributions with covariance $\sigma^2 I_{n \times n}$ and expectation having norm at most 1.

Spielman-Teng, Spielman-Deshpande, Vershynin, Dadush-Huiberts: There are $O(n^2 \sqrt{\log m}/\sigma^2)$ shadow vertices in expectation.

Tantalizing evidence from the simplex method

Consider a fixed two-dimensional plane $W \subset \mathbb{R}^n$. Let

$$P = \{x \in \mathbb{R}^n : \langle a, x \rangle \leq 1 \ \forall a \in A\}$$

where $A \subset \mathbb{R}^n, |A| = m$ is sampled independently from Gaussian distributions with covariance $\sigma^2 I_{n \times n}$ and expectation having norm at most 1.

Spielman-Teng, Spielman-Deshpande, Vershynin, Dadush-Huiberts:
There are $O(n^2\sqrt{\log m}/\sigma^2)$ shadow vertices in expectation.
The simplex method needs $O(n^2\sqrt{\log m}/\sigma^2)$ pivot steps in expectation to solve the LP.

Our results

Let

$$P = \{x \in \mathbb{R}^n : \langle a, x \rangle \leq 1 \ \forall a \in A\}$$

where $A \subset \mathbb{S}^{n-1}$ follows a Poisson point process with $\mathbb{E}[|A|] = m > 2^{\Omega(n)}$.

Our results

Let

$$P = \{x \in \mathbb{R}^n : \langle a, x \rangle \leq 1 \ \forall a \in A\}$$

where $A \subset \mathbb{S}^{n-1}$ follows a Poisson point process with $\mathbb{E}[|A|] = m > 2^{\Omega(n)}$.

Then with probability $1 - O(1/m)$ we have

$$\Omega(nm^{\frac{1}{n-1}}) \leq \text{Diameter(P)} \leq O(n^2 m^{\frac{1}{n-1}} + n^6 4^n).$$

The same holds for the shadow size on any fixed W.

Borgwardt: 34 years ago

Our results / This talk

Let

$$P = \{x \in \mathbb{R}^n : \langle a, x \rangle \leq 1 \ \forall a \in A\}$$

where $A \subset \mathbb{S}^{n-1}$ follows a Poisson point process with $\mathbb{E}[|A|] = m > 2^{\Omega(n)}$.

Then with probability $1 - O(1/m)$ we have

$$\Omega(nm^{\frac{1}{n-1}}) \leq \text{Diameter(P)} \leq O(n^2 m^{\frac{1}{n-1}} + n^6 4^n).$$

$$\Omega((m/\log m)^{\frac{1}{n-1}}) \leq \text{Diameter(P)} \leq O(m^{\frac{1}{n-1}} \cdot (\log(m) 3^n)^n).$$

The same holds for the shadow size on any fixed W.

Lower bound

$\varepsilon > 0$

A set $B \subset \mathbb{S}^{n-1}$ is called ε-dense if for every $x \in \mathbb{S}^{n-1}$ there exists $b \in B$ with $\|x - b\| \leq \varepsilon$.

Lower bound

A set $B \subset \mathbb{S}^{n-1}$ is called ε-dense if for every $x \in \mathbb{S}^{n-1}$ there exists $b \in B$ with $\|x - b\| \leq \varepsilon$.

→ A is $\underline{O((\log(m)/m)^{\frac{1}{n-1}})}{}^{=\varepsilon}$-dense with probability $1 - n^{-n}m^{-2}$.

Pick $N \subseteq S^{n-1}$ a minimal-size $\frac{\varepsilon}{n}$-dense set.

Using the union bound, prove that wph for every $p \in N$ there exists $a \in A$ such that $\|a - p\| \leq (1 - \frac{1}{n})\varepsilon$.

$|N| \leq \left(\frac{4n}{\varepsilon}\right)^{n-1} \leq (4n)^n \cdot m$

Lower bound

A set $B \subset \mathbb{S}^{n-1}$ is called ε-dense if for every $x \in \mathbb{S}^{n-1}$ there exists $b \in B$ with $\|x - b\| \leq \varepsilon$.

A is $O((\log(m)/m)^{\frac{1}{n-1}})$-dense with probability $1 - n^{-n}m^{-2}$.

$\varepsilon > 0$

$\rightarrow$ If A is ε-dense then every edge of P lies inside $(1-\varepsilon^2/2)^{-1}\mathbb{B}^n \setminus \mathbb{B}^n$.

$$\mathbb{B}^n \subseteq P \subseteq \frac{1}{1-\varepsilon^2/2}\mathbb{B}^n$$

$$P = \{x \in \mathbb{R}^n : \langle a, x \rangle \leq 1 \quad \forall a \in A\} \qquad A \subseteq S^{n-1}$$

pick $x \in P$ with $\|x\| \geq 1$. pick $a \in A$ such that $\|a - \frac{x}{\|x\|}\| \leq \varepsilon$.

$$\frac{1}{\|x\|} \geq \langle a, x/\|x\| \rangle = 1 - \|a - \tfrac{x}{\|x\|}\|^2/2 \geq 1 - \varepsilon^2/2$$

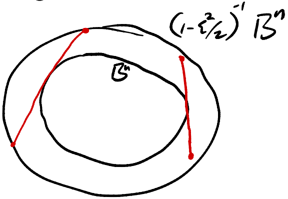
Lower bound

A set $B \subset \mathbb{S}^{n-1}$ is called ε-dense if for every $x \in \mathbb{S}^{n-1}$ there exists $b \in B$ with $\|x - b\| \leq \varepsilon$.

A is $O((\log(m)/m)^{\frac{1}{n-1}})$-dense with probability $1 - n^{-n}m^{-2}$.

If A is ε-dense then every edge of P lies inside $(1 - \varepsilon^2/2)^{-1}\mathbb{B}^n \setminus \mathbb{B}^n$.

If a line segment lies inside $(1 - \varepsilon^2/2)^{-1}\mathbb{B}^n \setminus \mathbb{B}^n$ and $\varepsilon \leq 1$ then it has length $\leq 2\varepsilon$



suppose line segment is $[v, v']$, $p \in [v, v']$ has minimum norm.

$$\|v - p\|^2 = \|v\|^2 + \|p\|^2 - 2\langle v, p\rangle$$
$$\leq \|v\|^2 - \|p\|^2 \leq \varepsilon.$$

Lower bound

A set $B \subset \mathbb{S}^{n-1}$ is called ε-dense if for every $x \in \mathbb{S}^{n-1}$ there exists $b \in B$ with $\|x - b\| \leq \varepsilon$.

A is $O((\log(m)/m)^{\frac{1}{n-1}})$-dense with probability $1 - n^{-n}m^{-2}$.

If A is ε-dense then every edge of P lies inside $(1 - \varepsilon^2/2)^{-1}\mathbb{B}^n \setminus \mathbb{B}^n$.

If a line segment lies inside $(1 - \varepsilon^2/2)^{-1}\mathbb{B}^n \setminus \mathbb{B}^n$ and $\varepsilon \leq 1$ then it has length $\leq 2\varepsilon$

If A is ε-dense then Diameter(P) $\geq \Omega(1/\varepsilon)$.

$v_1, \ldots, v_k \in P$ the vertices of a path with $\|v_1 - v_k\| \geq 1$.

$$1 \leq \|v_1 - v_k\| \leq \sum_{i=0}^{k-1} \|v_i - v_{i+1}\| \leq k \cdot 2\varepsilon \rightarrow k \geq \frac{1}{2\varepsilon}.$$

A weak upper bound

If $a, a' \in \mathbb{S}^{n-1}$ and $z \in \mathbb{R}^n$ satisfy $\|z\| \leq (1-\varepsilon^2/2)^{-1}$ and $\langle a, z\rangle, \langle a', z\rangle \geq 1$ then $\|a - a'\| \leq 2\varepsilon$.

$$1 - \varepsilon^2/2 \leq \frac{1}{\|z\|} \leq \langle a, z/\|z\| \rangle$$

$$= 1 - \|a - z/\|z\|\|^2/2.$$

$$\|a - a'\| \leq 2\varepsilon.$$

A weak upper bound

If $a, a' \in \mathbb{S}^{n-1}$ and $z \in \mathbb{R}^n$ satisfy $\|z\| \leq (1 - \varepsilon^2/2)^{-1}$ and $\langle a, z \rangle, \langle a', z \rangle \geq 1$ then $\|a - a'\| \leq 2\varepsilon$.

If A is ε-dense, $w_1, w_2 \in \mathbb{S}^{n-1}$ satisfy $\|w_1 - w_2\| \leq \varepsilon$, $z \in P$ is on the shadow path from w_1 to w_2, and $a \in A$ satisfies $\langle a, z \rangle = 1$ then $\|w_2 - a\| \leq 3\varepsilon$.

A weak upper bound

If $a, a' \in \mathbb{S}^{n-1}$ and $z \in \mathbb{R}^n$ satisfy $\|z\| \leq (1 - \varepsilon^2/2)^{-1}$ and $\langle a, z\rangle, \langle a', z\rangle \geq 1$ then $\|a - a'\| \leq 2\varepsilon$.

If A is ε-dense, $w_1, w_2 \in \mathbb{S}^{n-1}$ satisfy $\|w_1 - w_2\| \leq \varepsilon$, $z \in P$ is on the shadow path from w_1 to w_2, and $a \in A$ satisfies $\langle a, z\rangle = 1$ then $\|w_2 - a\| \leq 3\varepsilon$.

Every vertex on that shadow path is induced by n constraints from the set $A \cap \{x \in \mathbb{S}^{n-1} : \|w_2 - x\| \leq 3\varepsilon\}$.

A weak upper bound

If $a, a' \in \mathbb{S}^{n-1}$ and $z \in \mathbb{R}^n$ satisfy $\|z\| \le (1-\varepsilon^2/2)^{-1}$ and $\langle a, z\rangle, \langle a', z\rangle \ge 1$ then $\|a - a'\| \le 2\varepsilon$.

If A is ε-dense, $w_1, w_2 \in \mathbb{S}^{n-1}$ satisfy $\|w_1 - w_2\| \le \varepsilon$, $z \in P$ is on the shadow path from w_1 to w_2, and $a \in A$ satisfies $\langle a, z\rangle = 1$ then $\|w_2 - a\| \le 3\varepsilon$.

Every vertex on that shadow path is induced by n constraints from the set $A \cap \{x \in \mathbb{S}^{n-1} : \|w_2 - x\| \le 3\varepsilon\}$.

With probability $1 - 2n^{-n}m^{-2}$, A is $\varepsilon = O((m/\log(m))^{\frac{1}{n-1}})$-dense and every spherical cap of radius 3ε contains $\max_{x \in \mathbb{S}^{n-1}} |A \cap C(x, 3\varepsilon)| \le O(\log(m)3^n)$ points.

A weak upper bound

If $a, a' \in \mathbb{S}^{n-1}$ and $z \in \mathbb{R}^n$ satisfy $\|z\| \leq (1-\varepsilon^2/2)^{-1}$ and $\langle a, z\rangle, \langle a', z\rangle \geq 1$ then $\|a - a'\| \leq 2\varepsilon$.

If A is ε-dense, $w_1, w_2 \in \mathbb{S}^{n-1}$ satisfy $\|w_1 - w_2\| \leq \varepsilon$, $z \in P$ is on the shadow path from w_1 to w_2, and $a \in A$ satisfies $\langle a, z\rangle = 1$ then $\|w_2 - a\| \leq 3\varepsilon$.

Every vertex on that shadow path is induced by n constraints from the set $A \cap \{x \in \mathbb{S}^{n-1} : \|w_2 - x\| \leq 3\varepsilon\}$.

With probability $1 - 2n^{-n}m^{-2}$, A is $\varepsilon = O((m/\log(m))^{\frac{1}{n-1}})$-dense and every spherical cap of radius 3ε contains $\max_{x\in\mathbb{S}^{n-1}} |A \cap C(x, 3\varepsilon)| \leq O(\log(m)3^n)$ points.

By concatenating such paths; with probability $1 - 2n^{-n}m^{-2}$ every shadow path has length at most $(2\pi/\varepsilon)\cdot(\log(m)3^n)^n$.

Conclusion and future directions

With high probability we have

$$\underbrace{\Omega(nm^{\frac{1}{n-1}}) \leq \text{Diameter(P)} \leq O(n^2 m^{\frac{1}{n-1}} + n^6 4^n).}$$

The diameter is close to the expected shadow size, and a shadow size is close to its expectation.

Conclusion and future directions

With high probability we have

$$\Omega(nm^{\frac{1}{n-1}}) \leq \text{Diameter(P)} \leq O(n^2 m^{\frac{1}{n-1}} + n^6 4^n).$$

The diameter is close to the expected shadow size, and a shadow size is close to its expectation.

Does this hold for other probability distributions as well?