# On the applications of the Khinchine type inequality for Independent and Dependent Poisson random variables.

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Joint work with O. Herscovici and A.B. Kashlak

AGA seminar

# Poisson Distribution

Introduced by the Simèon-Denis Poisson in 1837.

The probability of k events occurring in an interval of unit time, which gives the Poisson distribution as

$$\mathbb{P}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 1, 2, \ldots, \tag{1}$$

where $0 < \lambda = \mathbb{E}(X) = Var(X)$.

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= \sum_{x_1}^{M} \frac{e^{-\lambda_1}\lambda_1^{x_1}}{x_1!} \frac{e^{-\lambda_2}\lambda_2^{M-x_1}}{(M-x_1)!}
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&= \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^M}{M!}.
\end{aligned}$$

For N independent Poisson distributed random variables $X_1, \dots, X_N$ their sum $\sum_{i=1}^{N} X_i \sim Poisson(\sum_{i=1}^{N} \lambda_i)$.

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$$\begin{aligned}
\mathbb{P}(\bar{X}|S = M) &= \frac{\mathbb{P}(\bar{X} \cap S = M)}{\mathbb{P}(S = M)} \\
&= \left(\prod_{i=1}^N \frac{e^{-\lambda_i}\lambda^{x_i}}{x_i!}\right) \Big/ \left(\frac{e^{-\sum_{i=1}^N \lambda_i}\left(\sum_{i=1}^N \lambda_i\right)^M}{M!}\right) \\
&= \binom{M}{x_1, x_2, \ldots, x_N} \prod_{i=1}^N \left(\frac{\lambda_i}{\sum_{i=1}^N \lambda_i}\right)^{x_i}
\end{aligned}$$

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$$= \left(\prod_{i=1}^{N} \frac{e^{-\lambda_i}\lambda^{x_i}}{x_i!}\right) / \left(\frac{e^{-\sum_{i=1}^{N}\lambda_i}\left(\sum_{i=1}^{N}\lambda_i\right)^M}{M!}\right)$$

$$= \binom{M}{x_1, x_2, \ldots, x_N} \prod_{i=1}^{N}\left(\frac{\lambda_i}{\sum_{i=1}^{N}\lambda_i}\right)^{x_i}$$

$$\sim Multinom\left(M; \frac{\lambda_1}{\sum_{i=1}^{N}\lambda_i}, \frac{\lambda_2}{\sum_{i=1}^{N}\lambda_i}, \ldots, \frac{\lambda_N}{\sum_{i=1}^{N}\lambda_i}\right).$$

It is hard to work with Multinomial distribution - this is the distribution function for discrete processes in which fixed probabilities prevail for each independently generated value.

# Khinchine Inequality

Let $a \in \mathbb{R}^N$.

$\varepsilon_i$, $i = 1, \ldots, N$ are independent Rademacher random variables, i.e.

$$P(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \frac{1}{2}, \text{ for } i = 1, \ldots, N.$$

**Khinchine Inequality**

For $q \geq 0$, there exists constant $K_q$, such that

$$\left( \mathbb{E} \left| \sum_{i=1}^{N} a_i \varepsilon_i \right|^q \right)^{\frac{1}{q}} \leq K_q \left\| a \right\|_2. \tag{2}$$

# Khinchine Type Inequality for different variables

- Continuous random variables uniformly distributed on symmetric intervals and random vectors uniformly distributed on the Euclidian spheres and balls (H. König, S. Kwapién, R. Latala, K, Oleszkiewicz);

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- Centered Gaussian random variables (A. Eskenazis, P. Nayar, T. Tkocz);
- Dependent random signs (O. Herscovici, A.B. Kashlak, S. Spektor).

# Goal

Let $X_i, i = 1, 2, \ldots, N$ be independent Poisson random variables with parameters $\lambda_i$ correspondingly. Let $a \in \mathbb{R}^N$. We would like to prove the following inequality:

$$\left( \mathbb{E} \left| \sum_{i=1}^{N} a_i X_i \right|^q \right)^{1/q} \leq C_q \left\| a \right\|_{\infty}. \tag{3}$$

Moreover, we are interested in the inequality (3) in the case when Poisson random variables are conditioned as following:

$$PO = \sum_{i=1}^{N} X_i = M \geq 0. \tag{4}$$

By $\mathbb{E}_{PO}$ we will be denoting expectation under assumption (4).

# Plan

- Tools
- Khinchine type inequality for independent Poisson random variables.
- Khinchine-type inequality under condition that the sum of the Poisson random variables is equal to some value $M$.
- Applications to bootstrap resampling in Big Data Sciences.

# Bell Polynomials

The incomplete exponential Bell polynomials:

$$\begin{aligned}
&\mathcal{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \\
&= \sum \frac{n!}{j_1! j_2! \cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},
\end{aligned} \tag{5}$$

here $j_1 + j_2 + \cdots + j_{n-k+1} = k$, $j_1 + 2j_2 + \cdots + (n-k+1)j_{n-k+1} = n$.

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here $j_1 + j_2 + \cdots + j_{n-k+1} = k$, $j_1 + 2j_2 + \cdots + (n-k+1)j_{n-k+1} = n$.
The $n$-th complete exponential Bell polynomial:

$$\mathcal{B}_n(x_1, \ldots, x_n) = \sum_{k=1}^{n} \mathcal{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \tag{6}$$

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The value of the Bell polynomial $\mathcal{B}_{n,k}(x_1, x_2, \ldots)$ on the sequence of ones equals a Stirling number of the second kind:

$$\mathcal{B}_{n,k}(1, 1, \ldots, 1) = S(n, k) = \left\{ {n \atop k} \right\}$$

# Bell Numbers

The sum of these values gives the value of the complete Bell polynomial on the sequence of ones:

$$B_n(1,1,\ldots,1) = \sum_{k=1}^{n} \mathcal{B}_{n,k}(1,1,\ldots,1) = \sum_{k=1}^{n} \left\{ {n \atop k} \right\},$$

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Bell numbers-upper bound (D. Berend, T. Tassa, 2010):

$$B_n < \left( \frac{0.792 \times n}{\ln(n+1)} \right)^n . \tag{7}$$

# Touchard Polynomials

If $X$ is a random variable with a Poisson distribution with expected value $\lambda$, then its $n$-th moment is $E(X^n) = T_n(\lambda)$, where

$$T_n(\lambda) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \lambda^k$$

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# Touchard Polynomials

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It can be expressed as the value of the complete Bell polynomial on all arguments being $\lambda$:

$$T_n(\lambda) = B_n(\lambda, \ldots, \lambda).$$

# Lambert $W$ function

$$W(x)e^{W(x)} = x. \tag{8}$$

The $W$ function has two real, and infinitely many complex branches.

Real branches:

$$W_0 : [-1/e, \infty) \to [-1, \infty)$$

and

$$W_{-1} : [-1/e, 0) \to (-\infty, -1].$$

Both of these are strictly monotone.

Special values:

$$W_0(0) = 0, W_0(e) = 1, W_0(-1/e) = -1, W_{-1}(-1/e) = -1.$$

# Khinchine type inequality for Independent Poisson Random variables

**Theorem (HKS, 2021)**

*Let $X_i, i = 1, \ldots, N$ be independent Poisson random variables with parameters $\lambda_i, i = 1, \ldots, N$ correspondingly. Let $a_i, i = 1, \ldots, N$ are in $\mathbb{R}$. Then, for positive integers $q$, we have*

$$\mathbb{E}\left|\sum_{i=1}^{N} a_i X_i\right|^q \leq T_q(\lambda_1 + \ldots + \lambda_N)\,\|a\|_\infty^q. \tag{9}$$

# Proof of Theorem

Using multinomial theorem,

$$\mathbb{E}\left|\sum_{i=1}^{N} a_i X_i\right|^q = \sum_{\substack{q_1+\ldots+q_N=q \\ q_i\in\{0,\ldots,q\}}} \frac{q!}{q_1!\ldots q_N!} a_1^{q_1}\ldots a_N^{q_N}\,\mathbb{E}\left(\prod_{i=1}^{N} X_i^{q_i}\right)$$

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$$= \sum_{\substack{q_1+\ldots+q_N=q \\ q_i\in\{0,\ldots,q\}}} \frac{q!}{q_1!\ldots q_N!} a_1^{q_1}\ldots a_N^{q_N} \prod_{i=1}^{N}\mathbb{E}(X_i^{q_i}).$$

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$$= \sum_{\substack{q_1+\ldots+q_N=q \\ q_i\in\{0,\ldots,q\}}} \frac{q!}{q_1!\ldots q_N!} a_1^{q_1}\ldots a_N^{q_N}\prod_{i=1}^{N}\mathbb{E}(X_i^{q_i}).$$

Consider

$$\prod_{i=1}^{N} a_i^{q_i} \le \prod_{i=1}^{N} |a_i|^{q_i} \le \prod_{i=1}^{N} \max_{1\le i\le N} |a_i|^{q_i} = \prod_{i=1}^{N} \|a\|_\infty^{q_i} = \|a\|_\infty^{q}.$$

Since $X_i$ are Poisson variables with means $\lambda_i$, $i = 1, \ldots, N$, the expectation $\mathbb{E}X_i^{q_i} = T_{q_i}(\lambda_i)$. Due to the independence of the variables, we have

$$\prod_{i=1}^{N} \mathbb{E}X_i^{q_i} = \prod_{i=1}^{N} T_{q_i}(\lambda_i).$$

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Altogether then,

$$\mathbb{E}\left|\sum_{i=1}^{N} a_i X_i\right|^q = \|a\|_\infty^q \sum_{\substack{q_1+\ldots+q_N=q \\ q_i\in\{0,\ldots,q\}}} \frac{q!}{q_1!\ldots q_N!} \prod_{i=1}^{N} T_{q_i}(\lambda_i)$$

$$= T_q(\lambda_1 + \ldots + \lambda_N)\|a\|_\infty^q. \quad \square$$

# Corollaries

## Corollary

*In case when* $\lambda_1 = \ldots = \lambda_N = \frac{1}{N}$, *we have*

$$\mathbb{E}\left|\sum_{i=1}^{N} a_i X_i\right|^q \leq B_q \|a\|_\infty^q < \left(\frac{0.792 \times q}{\ln(q+1)}\right)^q \|a\|_\infty^q, \tag{10}$$

Idea of the proof:

$$\prod_{i=1}^{N} \mathbb{E} X_i^{q_i} = \prod_{i=1}^{N} T_{q_i}(\lambda_i).$$

Use property $T_q(\lambda_1 + \ldots + \lambda_N) = T_q(1) = B_q$ and apply upper bound for Bell numbers.

## Corollary

*In case when* $\lambda_1 = \ldots = \lambda_N = 1$, *we have*

$$\mathbb{E}\left|\sum_{i=1}^{N} a_i X_i\right|^q \leq (1.15)^q \|a\|_\infty^q. \tag{11}$$

Idea of the proof:

$$\prod_{i=1}^{N} \mathbb{E} X_i^{q_i} = \prod_{i=1}^{N} T_{q_i}(\lambda_i).$$

Note, $\mathbb{E}\left(X^n\right) = B_n$.
We have,

$$\prod_{i=1}^{N} \mathbb{E} X_i^{q_i} = \prod_{i=1}^{N} B_{q_i} < \prod_{i=1}^{N} \left(\frac{0.792 q_i}{\ln(q_i + 1)}\right)^{q_i} < (1.15)^q.$$

## Corollary

*Let $a_i \in \mathbb{R}$ and $X_i, i = 1, \ldots, N$ be independent Poisson random variables with parameters $\lambda_1 = \ldots = \lambda_N = \lambda$. Then, for positive integers $q$,*

$$\mathbb{E}\left|\sum_{i=1}^{N} a_i X_i\right|^q \leq T_q(\lambda)\, N^{q/2} \|a\|_2^q. \tag{12}$$

*In particular, denoting $Z = \dfrac{1}{N}\sum_{i=1}^{N} \dfrac{a_i X_i}{\sigma}$, where $\bar{a} = \frac{1}{N}\sum_{i=1}^{N} a_i$ and $\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(a_i - \bar{a})^2$,*

$$\mathbb{E}\,|Z|^q \leq T_q(\lambda). \tag{13}$$

Idea of the proof is based on use of Hölder's inequality and logarithmic convexity of the Touchard Polynomials.

# Khinchine type inequality for Dependent Poisson Random variables

**Theorem (HKS, 2021)**

*Let $X_i$, $i = 1, \dots, N$ be positive Poisson random variables with parameters $\lambda_i$, $i = 1, \dots, N$ correspondingly, with additional condition that $PO = \sum_{i=1}^{N} X_i = M$. Let $a_i$, $i = 1, \dots, N$ are in $\mathbb{R}$. Then,*

$$\mathbb{E}_{PO}\left|\sum_{i=1}^{N} a_i X_i\right|^q \leq M^q \|a\|_\infty. \tag{14}$$

*and, under assumption that $X_i, i = 1, \dots, N$ are taking on values $\geq 1$,*

$$\mathbb{E}_{PO}\left|\sum_{i=1}^{N} a_i X_i\right|^q \leq (M - N + 1)^q \|a\|_1^q. \tag{15}$$

# Proof

**First bound:**

$$\mathbb{E}_{PO}\left|\sum_{i=1}^{N} a_i X_i\right|^q \leq \mathbb{E}_{PO}\left|\sum_{i=1}^{N} \max_{1\leq i\leq N} |a_i| X_i\right|^q$$

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**Second bound:**

Denote, $p_\alpha = \mathbb{P}(\prod_i X_i^{q_i} = \alpha)$.

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Denote, $p_\alpha = \mathbb{P}(\prod_i X_i^{q_i} = \alpha)$.

$$\mathbb{E}_{PO}\left(\prod_{i=1}^{N} X_i^{q_i}\right) = \sum_{\prod_i X_i^{q_i}=\alpha} \prod_i X_i^{q_i} p_\alpha$$

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Denote, $p_\alpha = \mathbb{P}(\prod_i X_i^{q_i} = \alpha)$.

$$\mathbb{E}_{PO}\left(\prod_{i=1}^{N} X_i^{q_i}\right) = \sum_{\prod_i X_i^{q_i}=\alpha} \prod_i X_i^{q_i} p_\alpha \leq \sum_{\prod_i X_i^{q_i}=\alpha} (\max_i X_i)^q p_\alpha = (\max_i X_i)^q.$$

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**Second bound:**

Denote, $p_\alpha = \mathbb{P}(\prod_i X_i^{q_i} = \alpha)$.

$$\mathbb{E}_{PO}\left(\prod_{i=1}^{N} X_i^{q_i}\right) = \sum_{\prod_i X_i^{q_i}=\alpha} \prod_i X_i^{q_i} p_\alpha \leq \sum_{\prod_i X_i^{q_i}=\alpha} (\max_i X_i)^q p_\alpha = (\max_i X_i)^q.$$

Since $X_i \in \mathbb{N}$, for any $i$ and $\sum_i^N X_i = M$, we have that $\max_i X_i = M - N + 1$. Therefore,

$$\mathbb{E}_{PO}\left|\sum_{i=1}^{N} a_i X_i\right|^q \leq (M-N+1)^q \left(\sum_{i=1}^{N} |a_i|\right)^q. \quad \square$$

## Corollary

*Let $a_i \in \mathbb{R}$ and $X_i, i = 1, \ldots, N$ be Poisson random variables with parameters $\lambda_1 = \ldots = \lambda_N = \lambda$ and such that $PO = \sum_{i=1}^{N} X_i = M$. Then, for positive integers $q$,*

$$\mathbb{E}_{PO} \left| \sum_{i=1}^{N} a_i X_i \right|^q \leq B_q \, N^{q/2} \|a\|_2^q. \tag{16}$$

*In particular, denoting $Z = \dfrac{1}{N} \sum_{i=1}^{N} \dfrac{a_i X_i}{\sigma}$, where $\bar{a} = \frac{1}{N} \sum_{i=1}^{N} a_i$ and $\sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (a_i - \bar{a})^2$,*

$$\mathbb{E} \, |Z|^q \leq B_q. \tag{17}$$

# Applications to the Bootstrap

**Bootstrapping** - powerful technique (Bradley Efron, 1979), used to construct Confidence Intervals (CI) for estimators with unknown distributions.

Given $N$ data points $a_1, \ldots, a_N \in \mathbb{R}$. It resamples the data with replacement by randomly drawing a new sample of size $N$ from the empirical distribution.

To construct a $(1-\alpha)$ bootstrapped confidence interval for the mean of the $a_i$:

- generate $D$ random resamples of the $a_i$ and compute the mean each time giving an ordered set $\bar{a}_1^* \leq \ldots \leq \bar{a}_D^*$.
- The bootstrapped confidence interval: $[\bar{a}_{\lfloor D\alpha/2 \rfloor}, \bar{a}_{\lceil D(1-\alpha/2) \rceil}]$, for $\alpha \in (0,1)$.

Advantages:

- Non-parametric statistical analysis.
- Easy construction of the CI from the resampling distribution.

Disadvantages:

- Computationally expensive.

Our results

$$\text{Poission:} \qquad \mathbb{E}\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N} a_i X_i\right|^q \leq T_q(\lambda)\,\|a\|_2^q,$$

$$\text{Multinomial:} \qquad \mathbb{E}_{PO}\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N} a_i X_i\right|^q \leq B_q\|a\|_2^q.$$

are used to construct a computation time-free approach of analytic formulae for bootstrap confidence intervals under Poisson and Multinomial resampling.

We also demonstrate an equivalence between these two resampling distributions.

# The Poisson Bootstrap

**Theorem (HKS, 2021)**

*Let $X_i, i = 1, \dots, N$ be independent Poisson random variables with parameters $\lambda_1 = \dots = \lambda_N = \lambda$. Let $a_i$ be i.i.d. nonnegative random variables with finite variance. Let $Z = \dfrac{1}{N}\sum_{i=1}^{N} \dfrac{a_i X_i}{\sigma}$, where $\bar{a} = \frac{1}{N}\sum_{i=1}^{N} a_i$ and $\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(a_i - \bar{a})^2$.*

*Then, for any $\alpha \in (0,1)$, $\lambda > -\ln(\alpha/2)$, the confidence interval for $Z$ is given by*

$$\mathbb{P}\left[\frac{\lambda + \ln(\alpha/2)}{W_{-1}\left(-\frac{1}{e}\left(1 + \frac{\ln(\alpha/2)}{\lambda}\right)\right)} \leq Z \leq \frac{-\lambda - \ln(\alpha/2)}{W_{-1}\left(-\frac{1}{e}\left(1 + \frac{\ln(\alpha/2)}{\lambda}\right)\right)}\right] \geq 1 - \alpha.$$

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$$\mathbb{P}(Z \geq t + \mu) = \mathbb{P}(e^{\nu Z} \geq e^{\nu(t+\mu)})$$

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&\leq e^{-\nu(t+\mu)}\mathbb{E}e^{\nu Z} \\
&\leq e^{-\nu(t+\mu)}e^{(\lambda(e^{\nu}-1))} \\
&\leq e^{-\lambda}\lambda^{t+\mu}(t+\mu)^{-(t+\mu)}e^{t+\mu}.
\end{aligned}$$

Last inequality comes form minimizing over all $\nu$:

$$0 = \frac{d}{d\nu}\left(-\nu(t+\mu) + \lambda(e^{\nu} - 1)\right) = -(t+\mu) + \lambda e^{\nu},$$

and $\nu = \ln\left(\frac{t+\mu}{\lambda}\right)$.

Markov-Chernoff's inequality, for any $\nu > 0$, $t > 0$:

$$\mathbb{P}(Y \geq t + \mu) \leq e^{-\nu(t+\mu)}\mathbb{E}e^{\nu Z} \leq e^{-\lambda}\lambda^{t+\mu}(t+\mu)^{-(t+\mu)}e^{t+\mu}.$$

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\ln(\alpha/2) &\geq e^u - \lambda + e^u\ln(\lambda) - e^u u \\
\lambda + \ln(\alpha/2) &\geq e^u(1 + \ln(\lambda) - u)
\end{aligned}$$

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The relation

$$e^{-u} \geq \frac{1 + \ln(\lambda) - u}{\lambda + \ln(\alpha/2)} \tag{18}$$

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$$u \leq 1 + \ln(\lambda) + W_{-1}\left(-\frac{1}{e} - \frac{\ln(\alpha/2)}{e\lambda}\right).$$

Since $t + \mu = e^u$, we get

$$t \leq \frac{-\lambda - \ln(\alpha/2)}{W_{-1}\left\{-\frac{1}{e}\left(1 + \frac{\ln(\alpha/2)}{\lambda}\right)\right\}} - \mu.$$

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# The Poisson Bootstrap

## Corollary

*Let $X_i, i = 1, \ldots, N$ be independent Poisson random variables with parameters $\lambda_1 = \ldots = \lambda_N = \lambda$. Let $a_i$ be i.i.d. nonnegative random variables with finite variance. Let $Z = \dfrac{1}{N}\sum_{i=1}^{N} \dfrac{a_i X_i}{\sigma}$, where $\bar{a} = \frac{1}{N}\sum_{i=1}^{N} a_i$ and $\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(a_i - \bar{a})^2$.*

*Then, for any $\alpha \in (0, 1)$, the confidence interval for $Z$ is given by*

$$\mathbb{P}\left(\sqrt{-2\lambda \ln\left(\frac{\alpha}{2}\right)} + \frac{\lambda \ln\left(\frac{\alpha}{2}\right)}{3} \leq Z \leq \sqrt{-2\lambda \ln\left(\frac{\alpha}{2}\right)} - \frac{\lambda \ln\left(\frac{\alpha}{2}\right)}{3}\right) \geq 1 - \alpha. \tag{19}$$

# The Multinomial Bootstrap

**Theorem (HKS, 2021)**

*Let $X_i, i = 1, \ldots, N$ be independent Poisson random variables with parameters $\lambda_1 = \ldots = \lambda_N = \lambda$ and such that $PO = \sum_{i=1}^N X_i = M$. Let $a_i$ be i.i.d. nonnegative random variables with finite variance. Let $Z = \frac{1}{N}\sum_{i=1}^N \frac{a_i X_i}{\sigma}$, where $\bar{a} = \frac{1}{N}\sum_{i=1}^N a_i$ and $\sigma^2 = \frac{1}{N}\sum_{i=1}^N (a_i - \bar{a})^2$.*

*Then, for any $\alpha \in (0,1)$, the confidence interval for $Z$ is given by*

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# Comparison of Poisson and Multinomial

We want to compare the confidence bounds from our Theorems with simulation-based Multinomial and Poisson bootstraps as well as with the standard parametric confidence interval based on the central limit theorem.

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Sampling regimes:

- $M = N$ for multinomial and $\lambda = 1$ for Poisson.
- $M = N/2$ for multinomial and $\lambda = 1/2$ for Poisson.

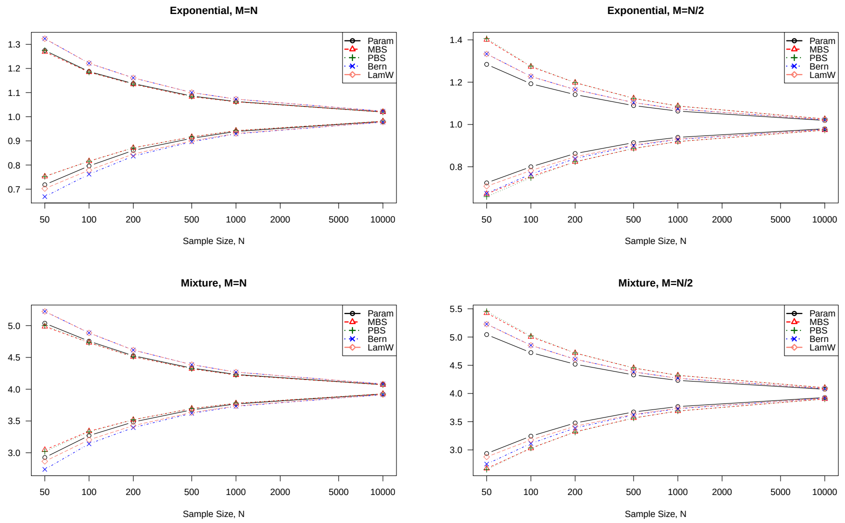


Figure: Simulated confidence upper and lower bounds for the four testing settings and five methods of interval construction.

# Asymptotics for Massive Data

There are situations where it would be impossible to fully resample a dataset such as for high throughput streaming data.

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The total sample size is set to $N = 1000000$, but the resampling size is only considered for $M = 50, 100, 1000$ or $\lambda = 0.00005, 0.0001, 0.001$.

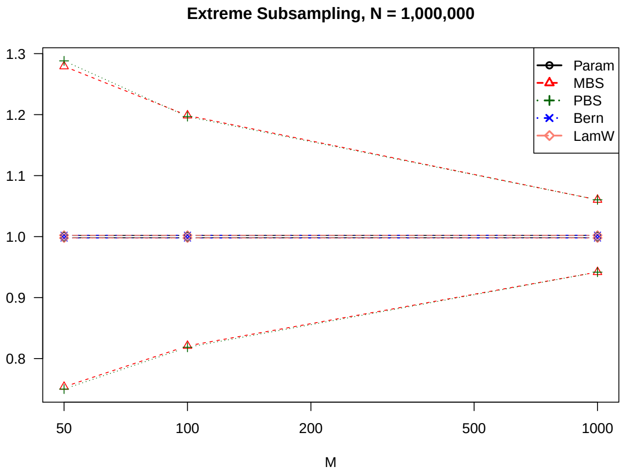


Figure: Simulated confidence upper and lower bounds in the extreme subsampling setting of $N = 1000000$ and $M = 50, 100, 1000$.

Thank You