On the fundamental gap and convex sets in hyperbolic space

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History of the Problem

- With T. Bourni, J. Clutterbuck, H. Nguyen, G. Wei, V. Wheeler.

- Introduction:
  - Given $\Omega \subset \mathbb{R}^n$ a domain (connected, compact, w. piecewise C^∞ boundary), the spectrum of the Laplacian $-\Delta$ (or, more general, $-\Delta + V$) with Dirichlet boundary condition

$$\begin{cases} \Delta u + \lambda u = 0, & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

    is discrete

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots < \lambda_n \to +\infty.$$

  - The fundamental gap of Ω is $\Gamma(\Omega) = \lambda_2 - \lambda_1$.
  - Ω can be a domain in any Riemannian manifold M.

History of the Problem

- Upper Bound on the Fundamental Gap:
  - Payne-Polya-Weinberger (PPW) Conjecture for (M^n, K), the n-dimensional simply connected space of constant curvature K:

For any bounded domain $\Omega \subset (M^n, K)$ (note that one does not need a convexity condition here),

$$\Gamma(\Omega) := \lambda_2(\Omega) - \lambda_1(\Omega) \leq \lambda_2(B_{\lambda_1}) - \lambda_1(B_{\lambda_1}),$$

where B_{λ_1} is a ball in (M^n, K) such that $\lambda_1(B_{\lambda_1}) = \lambda_1(\Omega)$.

The PPW conjecture was proved by Ashbaugh-Benguria ('92) for the case when $K = 0$, by Ashbaugh-Benguria ('01) for the case when $K = 1$ (and Ω in the upper hemisphere), and by Benguria-Linde ('07) for the case of $K = -1$.

Rayleigh Quotient of a Function w on Ω

$$\lambda_1 = \min_{Y_1} \frac{\int_\Omega |\nabla w|^2 dx}{\int_\Omega w^2 dx} \quad \text{and} \quad \lambda_n = \min_{Y_n} \frac{\int_\Omega |\nabla w|^2 dx}{\int_\Omega w^2 dx},$$

$Y_1 = \{w \in \mathcal{C}^2(\Omega) \mid w \not\equiv 0 \text{ on } \Omega, w = 0 \text{ on } \partial\Omega\}$

$Y_n = \{w \in \mathcal{C}^2(\Omega) \mid w \not\equiv 0 \text{ on } \Omega, w = 0 \text{ on } \partial\Omega, \;\; < w, v_i >= 0\}$
where $v_1, \ldots, v_{n-1}$ are the first $(n-1)$ eigenfunctions.

Idea for proving PPW: Use a smooth function $P \not\equiv 0$ with $\int_\Omega P v_1^2 \, dx = 0$ and

$$\lambda_2 - \lambda_1 \leq \frac{\int_\Omega |\nabla P|^2 v_1^2 \, dx}{\int_\Omega P^2 v_1^2 \, dx}.$$

History of the Problem in $\mathbb{R}^n$

- Lower Bound on the Fundamental Gap:

For bounded convex domain Ω in $\mathbb{R}^n$ and Schrödinger operator $-\Delta + V$, where $V \geq 0$, convex, the question carries a special name:

- The Fundamental Gap Conjecture (van der Berg '83, Yau '86, Ashbaugh-Benguria '89)

$$\Gamma(\Omega, V) := \lambda_2(\Omega, V) - \lambda_1(\Omega, V) \geq \frac{3\pi^2}{D^2}, \quad \text{where } D := \text{Diam}\,(\Omega).$$

- The lower bound is approached when $V = 0$, and the domain is a degenerating thin rectangular box.

On the Fundamental Gap Conjecture in $\mathbb{R}^n$

- Progress was slow for a while even in the 1-dimensional case, see early survey by Ashbaugh at AIM, May 2006.
- Singer-Wong-Yau-Yau established the bound $\Gamma(\Omega, V) \geq \frac{\pi^2}{4D^2}$ using the so-called Li-Yau gradient method, '85;
- later improved by Yu-Zhong, and Ling, '86 to $\Gamma(\Omega, V) \geq \frac{\pi^2}{D^2}$. NOTE: Key to the SWYY breakthrough was log-concavity of the groundstate of Schrödinger operators with convex potential, [Ashbaugh], earlier work of Brascamp and Lieb.
- Further improvement by Yau '03 depending on upper bound estimate on the log-concavity of the first eigenfunction.
- The Fundamental Gap Conjecture is solved with new ideas such as the modulus of continuity and concavity by Andrews-Clutterbuck '11. Proofs w/o heat equation followed.

Fundamental Gap on $\mathbb{S}^n$

- In 2018, 2019, Dai, He, Seto, Wang, and Wei (in various subsets) generalized the fundamental gap of estimate of $-\Delta$ with Dirichlet boundary conditions to convex domains in $\mathbb{S}^n$, showing that

$$\Gamma(\Omega) := \lambda_2(\Omega) - \lambda_1(\Omega) \geq \frac{3\pi^2}{D^2},$$

- with same degenerating case giving the bound.
- It is natural to ask if this holds also for convex domains in $\mathbb{H}^n$.

Similarities and Differences

- In both settings, the log-concavity of the first eigenfunction plays an important role.
- Just log-concavity is sufficient to obtain $\lambda_2 - \lambda_1 \geq \frac{\pi^2}{D^2}$, but to obtain the optimal estimates it is shown that the first eigenfunction is *super log-concave*, i.e. more log-concave than the first eigenfunction of the one-dimensional model operator,

$$L_{n,K,D}(\phi) = \phi'' - (n-1)\mathrm{tn}_K(s)\phi' \tag{1}$$

on $[-\frac{D}{2}, \frac{D}{2}]$ with Dirichlet boundary condition. Here

$$\mathrm{tn}_K(s) = \begin{cases} \sqrt{K}\tan(\sqrt{K}s), & K > 0 \\ 0, & K = 0 \\ -\sqrt{-K}\tanh(\sqrt{-K}s) & K < 0 \end{cases}$$

where $K = 0$ is the model for $\mathbb{R}^n$ & $K = 1$ is the model for $\mathbb{S}^n$.
- $K = -1$ is not a good model for $\mathbb{H}^n$, though the first eigenfunction of (1) when $K = -1$ is still log-concave.

Main Results

Theorem (BCNSWW)

There are convex domains in $\mathbb{H}^2$ *such that*

$$\lambda_2 - \lambda_1 < 3\pi^2/D^2,$$

where D *is the diameter of the domain.*

Theorem (BCNSWW)

For any $n \geq 2$, *any* $\epsilon > 0$, *and any* $D > 0$ *fixed real number, there exist convex domains in* $\mathbb{H}^n$ *such that*

$$\lambda_2 - \lambda_1 < \epsilon\pi^2/D^2,$$

where D *is the diameter of the domain.*

On the Convex Domains

Let $\mathbb{H}^2$ be the hyperbolic space modelled by the Poincaré half-plane $\{(x, y) \mid y > 0\} = \{(r, \theta) \mid r > 0, \theta \in (0, \pi)\}$ with the metric

$$g = ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{dr^2}{r^2 \sin^2 \theta} + \frac{d\theta^2}{\sin^2 \theta}.$$

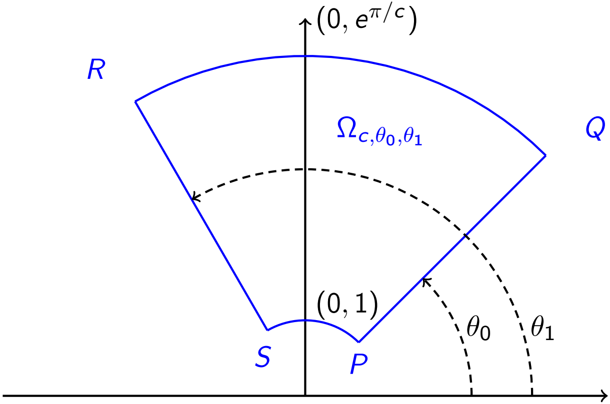
In the orthonormal frame $e_1 = r \sin\theta \frac{\partial}{\partial r}$, $e_2 = \sin\theta \frac{\partial}{\partial \theta}$, the non-vanishing Christoffel symbols are $\Gamma_{11}^2 = -\Gamma_{12}^1 = -\Gamma_{21}^1 = \cos\theta$, so the Laplace operator becomes

$$\Delta v = v_{11} + v_{22} = r^2 \sin^2\theta \ v_{rr} + \sin^2\theta \ v_{\theta\theta} + r \sin^2\theta \ v_r.$$

On the Convex Domains

We consider the family of domains

$$\Omega_{c,\theta_0,\theta_1} = \{(r,\theta) \mid 1 < r < e^{\pi/c},\ \theta_0 < \theta < \theta_1\}.$$



More on the domains

- Why this type of domains?
- The following inequalities holds for the diameter $D := D_{c,\theta_0,\theta_1}$:

  $$\operatorname{arccosh}\left(\csc\theta_* \cosh\frac{\pi}{c}\right) \le D \le \operatorname{arccosh}\left(\csc^2\theta_* \cosh\frac{\pi}{c} + \cot^2\theta_*\right),$$

  where $\theta_* := \min\{\theta_0, \pi - \theta_1\}$.

  Right inequality holds because our domain is included in $\Omega_{c,\theta_*,\pi-\theta_*}$ and r.h.s. is its diameter.
  Left inequality reflects that the diameter is greater than the distance from P (or R) to the point $(0, e^{\pi/c})$.
- The following limit holds: $\frac{\pi^2}{c^2 D^2_{c,\theta_0,\theta_1}} \to 1$ as $c \to 0$ or $\theta_* \to \frac{\pi}{2}$.

The Eigenvalues

- Setting $u(r,\theta) = f(r)\,h(\theta)$, $\Delta u = -\lambda u$ gives

$$r^2 \sin^2\theta \; f_{rr}\, h + r \sin^2\theta \; f_r\, h + \sin^2\theta \; f \; h_{\theta\theta} = -\lambda f h,$$
$$\left(r^2 \frac{f_{rr}}{f} + r\frac{f_r}{f}\right) + \left(\frac{h_{\theta\theta}}{h} + \lambda \csc^2\theta\right) = 0.$$

- Hence we should solve the two eigenvalue equations

$$r^2 f_{rr} + r f_r = -\mu f, \quad r \in [1, e^{\pi/c}], \tag{2}$$
$$h_{\theta\theta} + \lambda \csc^2\theta\, h = \mu h, \quad \theta \in [\theta_0, \theta_1], \tag{3}$$

both with Dirichlet conditions. Let $t = \log r$, so (2) becomes

$$f_{tt} = -\mu f, \quad t \in [0, \tfrac{\pi}{c}]. \tag{4}$$

- In order for this to satisfy the boundary conditions, μ must be positive, so we set $\mu = (kc)^2, f(t) = \sin(kct)$, where $k \in \mathbb{Z}^*$.

The First Eigenvalue

- The first Dirichlet eigenvalue λ_1 of $\Delta u = -\lambda u$ on $\Omega_{c,\theta_0,\theta_1}$ corresponds to a strictly positive eigenfunction, which implies that f in (4) is $\sin(ct)$, and so $\mu = c^2$ and λ_1 is given by the value λ solving

$$\begin{aligned} h_{\theta\theta} + \lambda \csc^2\theta\, h = c^2 h, \quad \theta \in [\theta_0, \theta_1], \\ h(\theta_0) = h(\theta_1) = 0, \end{aligned} \tag{5}$$

for $h > 0$. We denote by $\lambda_1 = \lambda_1^{c^2}$ the smallest such λ.

The Second Eigenvalue

- The second eigenvalue λ_2 corresponds to a sign-changing eigenfunction: either f or h changes sign. If f changes sign, then f in (4) is given by $\sin(2ct)$ and $\mu = 4c^2$; in this case λ_2 is given by $\lambda_1^{4c^2}$ solving

$$h_{\theta\theta} + \lambda \csc^2 \theta \, h = 4c^2 h, \quad \theta \in [\theta_0, \theta_1], \tag{6}$$

  with $h > 0$ and Dirichlet boundary conditions. Otherwise h changes sign, f is positive and is given by $\sin(ct)$ with $\mu = c^2$; then λ_2 is given by $\lambda_2^{c^2}$ solving (5) with h changing sign exactly once.

- Thus, the second eigenvalue is $\lambda_2 = \min\{\lambda_1^{4c^2}, \lambda_2^{c^2}\}$.

Estimating the Eigenvalues

- The first eigenvalue of (3): $h_{\theta\theta} + \lambda \csc^2\theta\, h = \mu h$ denoted by λ_1^μ, satisfies

$$\sin^2(\theta_*)\left(\mu + \frac{\pi^2}{(\theta_1-\theta_0)^2}\right) \le \lambda_1^\mu \le \mu + \frac{\pi^2}{(\theta_1-\theta_0)^2}. \qquad (7)$$

- **Proof:** Let h be a solution of (3). We multiply both sides of the equation by h and integrate, to obtain

$$\lambda = \frac{\int_{\theta_0}^{\theta_1}(|h_\theta|^2 + \mu h^2)d\theta}{\int_{\theta_0}^{\theta_1}(\csc^2\theta)h^2 d\theta} > \frac{1}{\csc^2(\theta_*)}\left(\mu + \frac{\int_{\theta_0}^{\theta_1}|h_\theta|^2 d\theta}{\int_{\theta_0}^{\theta_1}h^2 d\theta}\right)$$

and use Wirtinger's inequality $\int_0^D (h')^2\,dx \ge (\pi/D)^2\int_0^D h^2\,dx$. From above, choose the test function $\varphi = \sin\left(\frac{\theta-\theta_0}{\theta_1-\theta_0}\pi\right)$ in Rayleigh quotient. But $\csc^2\theta \ge 1$, so $\lambda_1^\mu \le \mu + \dfrac{\pi^2}{(\theta_1-\theta_0)^2}$.

Then the following estimate for λ_2^μ, the second eigenvalue of (3) holds:

$$\sin^2\theta_*\left(\mu+\frac{4\pi^2}{(\theta_1-\theta_0)^2}\right)\le\lambda_2^\mu\le\mu+\frac{4\pi^2}{(\theta_1-\theta_0)^2} \qquad (8)$$

Combining (8) and (7), we have $\lambda_2^{c^2}\ge\lambda_1^{4c^2}$ when

$$\frac{\pi^2}{(\theta_1-\theta_0)^2}\frac{(4\sin^2\theta_*-1)}{(4-\sin^2\theta_*)}\ge c^2,\quad \theta_*>\frac{\pi}{6}. \qquad (9)$$

Lemma

Assume that c, θ_* satisfies (9). Then the fundamental gap of $\Omega_{c,\theta_0,\theta_1}$ satisfies

$$3\sin^2\theta_* c^2<\lambda_2-\lambda_1<3c^2. \qquad (10)$$

Hence, as θ_* approaches $\frac{\pi}{2}$, the gap approaches $3c^2$.

Application of Sturm's Comparison Theorem

In fact,

$$\sin^2\theta_*[1+\frac{c}{\pi}\ln(\csc\theta_*)]^2 < (\lambda_2-\lambda_1)\frac{D^2_{c,\theta_0,\theta_1}}{3\pi^2} < \left[1+\frac{c}{\pi}\left(2\ln(\csc\theta_*)+\eta\right)\right]^2$$

where $\eta := \eta(c,\theta_*) > 0$ and goes to zero if $\theta_* \nearrow \pi/2$ or $c \searrow 0$.

The left hand side is ≥ 1 when $c > \pi$ and $\sin\theta_* \geq \exp(\frac{\pi}{c}-1)$. For any fixed $c > \pi$, and for all θ_* sufficiently close to $\frac{\pi}{2}$, we have that $c,\ \theta_*$ satisfy $\sin\theta_* \geq \exp(\frac{\pi}{c}-1)$ and condition (9).

For these domains, the gap satisfies the inequality $\lambda_2-\lambda_1 > \frac{3\pi^2}{D^2}$.

- Consider a family of problems generalizing (3), indexed by a parameter t

$$h'' + v(t)h = \mu(t)h \text{ on } I = [\theta_0, \theta_1], \tag{11}$$

  with vanishing Dirichlet boundary conditions. Here $h(\theta) = h^t(\theta)$ depends on t, and v as well, $v(t) = \lambda(t)\csc^2\theta$. Let $\lambda(t)$ be the first eigenvalue for each t, which is smooth in t, and $h^t(\theta)$ are all first eigenfunctions, so $h^t(\theta) > 0$ on (θ_0, θ_1).
- Denoting derivatives with respect to t as $\dot{h}$, we get

$$\dot{h}'' + \dot{v}h + v\dot{h} = \dot{\mu}h + \mu\dot{h} \text{ on } I. \tag{12}$$

- To relate changes in μ with changes in v, we multiply (12) by h, integrate over I, and use (11) to find

$$\dot{\mu}\int h^2 d\theta = \int \dot{v}h^2 d\theta.$$

- Therefore, if we set $\mu(t) = c^2 + 3c^2 t$, we have

$$3c^2 \int h^2 d\theta = \int \dot{\lambda}(\csc^2\theta) h^2 d\theta = \dot{\lambda} \int (\csc^2\theta) h^2 d\theta.$$

  Rearrange

$$\dot{\lambda} = 3c^2 \frac{\int (h^t)^2 d\theta}{\int (\csc^2\theta)(h^t)^2 d\theta},$$

  and integrate from $t = 0$ to 1, recalling $\lambda(0) = \lambda_1$, $\lambda(1) = \lambda_2$:

- Proposition

$$\lambda_2 - \lambda_1 \leq 3c^2 \max_t \frac{\int (h^t)^2 d\theta}{\int (\csc^2\theta)(h^t)^2 d\theta} \leq 3c^2(1 - \delta), \tag{13}$$

for some $\delta = \delta(\theta_0, \theta_1) > 0$, independent of c.

  - Combining with the estimate on diameter, we get that for all c sufficiently small, and $\theta_* > \frac{\pi}{6}$, the fundamental gap of the domains $\Omega_{c,\theta_0,\theta_1}$ is less than $3\pi^2/D^2$.

Conclusions

At the talk (-: