On the unique determination of ellipsoids by dual volumes

Kateryna Tatarko

(joint work with S. Myroschnychenko and V. Yaskin)

Online Asymptotic Geometric Analysis Seminar

Classical Steiner formula

For a convex body K and $t > 0$,

$$\mathrm{vol}_n(K + tB_2^n) = \sum_{i=0}^{n} \kappa_{n-i} V_i(K) t^{n-i}$$

where κ_{n-i} is the volume of B_2^{n-i}. The coefficients $V_i(K)$ are called *intrinsic volumes*.

In particular,

$$\begin{aligned} V_0(K) &= \text{the volume of the unit ball;} \\ V_1(K) &= \text{(multiple of) the mean width;} \\ V_{n-1}(K) &= \text{(multiple of) the surface area;} \\ V_n(K) &= \text{the volume of } K. \end{aligned}$$

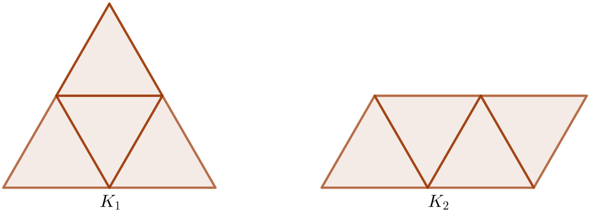
Intrinsic volumes and uniqueness of convex bodies

Question: *Can a convex body be determined by its intrinsic volumes?*

Intrinsic volumes and uniqueness of convex bodies

Question: *Can a convex body be determined by its intrinsic volumes?*

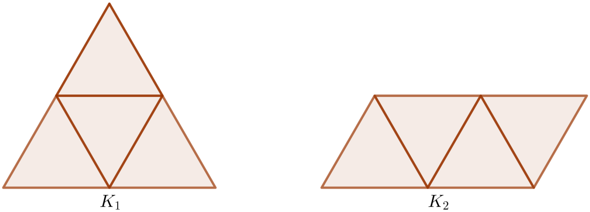
In general, no.



Intrinsic volumes and uniqueness of convex bodies

Question: Can a convex body be determined by its intrinsic volumes?

In general, no.



To expect positive answer: consider some *n*-parametric families of convex bodies.

- rectangular parallelepipeds: yes (Vietta theorem);

Ellipsoids

- ellipsoids:

$$\mathcal{E}_a = \left\{ x \in \mathbb{R}^n : \frac{x_1^2}{a_1^2} + \cdots + \frac{x_n^2}{a_n^2} \leq 1 \right\}.$$

Conjecture (Gusakova, Zaporozhets (2017))

Let $\mathcal{E}_a$ and $\mathcal{E}_b$ be two ellipsoids in $\mathbb{R}^n$ such that $V_1(\mathcal{E}_a) = V_1(\mathcal{E}_b)$, $V_2(\mathcal{E}_a) = V_2(\mathcal{E}_b)$,..., $V_n(\mathcal{E}_a) = V_n(\mathcal{E}_b)$. Then $\mathcal{E}_a$ and $\mathcal{E}_b$ are congruent.

$\mathbb{R}^2$: true;

$\mathbb{R}^3$: true, Petrov, Tarasov (2020);

$\mathbb{R}^n$, $n \geq 4$: problem is open.

Steiner formula of the dual Brunn Minkowski theory

For a star body K in $\mathbb{R}^n$ and $t > 0$, we have that

$$\mathrm{vol}_n(K \,\widetilde{+}\, tB_2^n) = \sum_{i=0}^{n} \binom{n}{i} \widetilde{V}_i(K) t^{n-i},$$

where $\widetilde{+}$ is a radial addition. The coefficients $\widetilde{V}_i(K)$ are called *dual volumes*.

Steiner formula of the dual Brunn Minkowski theory

For a star body K in $\mathbb{R}^n$ and $t > 0$, we have that

$$\mathrm{vol}_n(K \widetilde{+} tB_2^n) = \sum_{i=0}^{n} \binom{n}{i} \widetilde{V}_i(K) t^{n-i},$$

where $\widetilde{+}$ is a radial addition. The coefficients $\widetilde{V}_i(K)$ are called *dual volumes*.

Dual volumes of order i can be written as

$$\widetilde{V}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^i(\theta) d\theta,$$

where $\rho_K(\theta) = \max\{\lambda \geq 0 \,|\, \lambda\theta \in K\}$ is the radial function of K.

Dual question

Question: *Can an ellipsoid be determined by its dual volumes?*

Dual question

Question: *Can an ellipsoid be determined by its dual volumes?*

Yes!

Theorem (Myroshnychenko, T., Yaskin (2020))

Let $\mathcal{E}_a$ and $\mathcal{E}_b$ be two ellipsoids in $\mathbb{R}^n$, $n \geq 2$, centered at the origin such that $\widetilde{V}_1(\mathcal{E}_a) = \widetilde{V}_1(\mathcal{E}_b)$, $\widetilde{V}_2(\mathcal{E}_a) = \widetilde{V}_2(\mathcal{E}_b)$,..., $\widetilde{V}_n(\mathcal{E}_a) = \widetilde{V}_n(\mathcal{E}_b)$. Then $\mathcal{E}_a$ and $\mathcal{E}_b$ are congruent.

Representation of dual volumes

Lemma

Let $\mathcal{E}_a$ be an ellipsoid centered at the origin with semi-axes $a_1, \dots, a_n$. If $i \in \mathbb{R}$ such that $0 < i < n$, then

$$\widetilde{V}_i(\mathcal{E}_a) = \frac{4\pi^{n/2}}{n\Gamma\left(\frac{n-i}{2}\right)\Gamma\left(\frac{i}{2}\right)} \int_0^\infty \frac{u^{i-1}}{\sqrt{(1+\frac{u^2}{a_1^2})\cdots(1+\frac{u^2}{a_n^2})}}\, du.$$

Representation of dual volumes

Lemma

Let $\mathcal{E}_a$ be an ellipsoid centered at the origin with semi-axes $a_1, \ldots, a_n$. If $i \in \mathbb{R}$ such that $0 < i < n$, then

$$\widetilde{V}_i(\mathcal{E}_a) = \frac{4\pi^{n/2}}{n\Gamma\left(\frac{n-i}{2}\right)\Gamma\left(\frac{i}{2}\right)} \int_0^{\infty} \frac{u^{i-1}}{\sqrt{(1+\frac{u^2}{a_1^2})\cdots(1+\frac{u^2}{a_n^2})}}\, du.$$

Sketch of the proof:

Recall that

$$\widetilde{V}_i(\mathcal{E}_a) = \frac{1}{n}\int_{S^{n-1}} \rho^i_{\mathcal{E}_a}(\theta)d\theta = \frac{1}{n}\int_{S^{n-1}} \|\theta\|^{-i}_{\mathcal{E}_a}\, d\theta,$$

where $\|\theta\|_{\mathcal{E}_a}$ is the Minkowski functional of $\mathcal{E}_a$, i.e.

$$\|x\|_{\mathcal{E}_a} = \left(\frac{x_1^2}{a_1^2} + \cdots + \frac{x_n^2}{a_n^2}\right)^{1/2}, \qquad x \in \mathbb{R}^n.$$

Representation of dual volumes

For $i \in (0, n)$, we have

$$\int_{S^{n-1}} \|\theta\|_{\mathcal{E}_a}^{-i} d\theta = \frac{2}{\Gamma\left(\frac{n-i}{2}\right)} \int_{\mathbb{R}^n} \|x\|_{\mathcal{E}_a}^{-i} e^{-|x|^2}\, dx.$$

(pass to polar coordinates in the latter integral)

Representation of dual volumes

For $i \in (0, n)$, we have

$$\int_{S^{n-1}} \|\theta\|_{\mathcal{E}_a}^{-i} d\theta = \frac{2}{\Gamma\left(\frac{n-i}{2}\right)} \int_{\mathbb{R}^n} \|x\|_{\mathcal{E}_a}^{-i} e^{-|x|^2}\, dx.$$

(pass to polar coordinates in the latter integral)

Note that

$$\|x\|_{\mathcal{E}_a}^{-i} = \frac{2}{\Gamma\left(\frac{i}{2}\right)} \int_0^{\infty} u^{i-1} e^{-\|x\|_{\mathcal{E}_a}^2 u^2}\, du.$$

(change of variables $s = \|x\|_{\mathcal{E}_a} u$ in the integral)

Representation of dual volumes

For $i \in (0, n)$, we have

$$\int_{S^{n-1}} \|\theta\|_{\mathcal{E}_a}^{-i} d\theta = \frac{2}{\Gamma\left(\frac{n-i}{2}\right)} \int_{\mathbb{R}^n} \|x\|_{\mathcal{E}_a}^{-i} e^{-|x|^2}\, dx.$$

(pass to polar coordinates in the latter integral)

Note that

$$\|x\|_{\mathcal{E}_a}^{-i} = \frac{2}{\Gamma\left(\frac{i}{2}\right)} \int\limits_0^\infty u^{i-1} e^{-\|x\|_{\mathcal{E}_a}^2 u^2}\, du.$$

(change of variables $s = \|x\|_{\mathcal{E}_a} u$ in the integral)

Thus,

$$\widetilde{V}_i(\mathcal{E}_a) = \frac{4}{n\Gamma\left(\frac{n-i}{2}\right)\Gamma\left(\frac{i}{2}\right)} \int\limits_0^\infty u^{i-1} \int_{\mathbb{R}^n} e^{-\left(\frac{x_1^2}{a_1^2}+\cdots+\frac{x_n^2}{a_n^2}\right) u^2} e^{-|x|^2}\, dx\, du$$

Representation of dual volumes

$$= \frac{4}{n\Gamma\left(\frac{n-i}{2}\right)\Gamma\left(\frac{i}{2}\right)} \int_0^\infty u^{i-1} \int_{\mathbb{R}} e^{-x_1^2\left(1+\frac{u^2}{a_1^2}\right)}\, dx_1 \cdots \int_{\mathbb{R}} e^{-x_n^2\left(1+\frac{u^2}{a_n^2}\right)}\, dx_n\, du$$

Representation of dual volumes

$$= \frac{4}{n\Gamma\left(\frac{n-i}{2}\right)\Gamma\left(\frac{i}{2}\right)} \int_0^\infty u^{i-1} \int_{\mathbb{R}} e^{-x_1^2\left(1+\frac{u^2}{a_1^2}\right)} dx_1 \cdots \int_{\mathbb{R}} e^{-x_n^2\left(1+\frac{u^2}{a_n^2}\right)} dx_n \, du$$

$$= \frac{4}{n\Gamma\left(\frac{n-i}{2}\right)\Gamma\left(\frac{i}{2}\right)} \int_0^\infty u^{i-1} \frac{\sqrt{\pi}}{\sqrt{1+\frac{u^2}{a_1^2}}} \cdots \frac{\sqrt{\pi}}{\sqrt{1+\frac{u^2}{a_n^2}}} \, du.$$

(use $\int_{\mathbb{R}} e^{-y^2 t} dy = \frac{\sqrt{\pi}}{\sqrt{t}}$ for $t > 0$)

Sketch of the proof (main theorem) I

Now we give the idea for the proof of the main theorem.

$$\widetilde{V}_i(\mathcal{E}_a) = \widetilde{V}_i(\mathcal{E}_b), \ i = 1, \dots, n \quad \Rightarrow \quad \begin{cases} \int\limits_0^\infty \frac{u^{i-1}}{\sqrt{(1+\frac{u^2}{a_1^2})\cdots(1+\frac{u^2}{a_n^2})}}\, du = \int\limits_0^\infty \frac{u^{i-1}}{\sqrt{(1+\frac{u^2}{b_1^2})\cdots(1+\frac{u^2}{b_n^2})}}\, du, \\ \qquad i = 1, \dots, n-1; \\ a_1 \cdots a_n = b_1 \cdots b_n. \end{cases}$$

Sketch of the proof (main theorem) I

Now we give the idea for the proof of the main theorem.

$$\widetilde{V}_i(\mathcal{E}_a) = \widetilde{V}_i(\mathcal{E}_b), \;\; i = 1, \dots, n \quad \Rightarrow \quad \begin{cases} \int\limits_0^\infty \frac{u^{i-1}}{\sqrt{(1+\frac{u^2}{a_1^2})\cdots(1+\frac{u^2}{a_n^2})}}\, du = \int\limits_0^\infty \frac{u^{i-1}}{\sqrt{(1+\frac{u^2}{b_1^2})\cdots(1+\frac{u^2}{b_n^2})}}\, du, \\ \qquad i = 1, \dots, n-1; \\ a_1 \cdots a_n = b_1 \cdots b_n. \end{cases}$$

Denote

$$F(u) = \frac{1}{\sqrt{(1+\frac{u^2}{a_1^2})\cdots(1+\frac{u^2}{a_n^2})}} - \frac{1}{\sqrt{(1+\frac{u^2}{b_1^2})\cdots(1+\frac{u^2}{b_n^2})}}.$$

Thus,

$$\int\limits_0^\infty u^{i-1} F(u)\, du = 0$$

for $i = 1, \dots, n-1$.

Sketch of the proof (main theorem) II

This implies that for any polynomial $P(u)$ of degree at most $n-2$, we have

$$\int_0^\infty P(u)F(u)\,du = 0;$$

Sketch of the proof (main theorem) II

This implies that for any polynomial $P(u)$ of degree at most $n-2$, we have

$$\int_0^\infty P(u)F(u)\,du = 0;$$

Now, our goal is to show that F has at most $n-2$ positive real roots.

If so, we can choose $P(u)$ such that it changes sign exactly at these positive numbers

Sketch of the proof (main theorem) II

This implies that for any polynomial $P(u)$ of degree at most $n-2$, we have

$$\int_0^\infty P(u)F(u)\,du = 0;$$

Now, our goal is to show that F has at most $n-2$ positive real roots.

If so, we can choose $P(u)$ such that it changes sign exactly at these positive numbers

$\Rightarrow$ PF is either everywhere positive or everywhere negative;

Sketch of the proof (main theorem) II

This implies that for any polynomial $P(u)$ of degree at most $n-2$, we have

$$\int_0^\infty P(u)F(u)\,du = 0;$$

Now, our goal is to show that F has at most $n-2$ positive real roots.

If so, we can choose $P(u)$ such that it changes sign exactly at these positive numbers

$\Rightarrow$ PF is either everywhere positive or everywhere negative;

$\Rightarrow$ $F \equiv 0$;

Sketch of the proof (main theorem) II

This implies that for any polynomial $P(u)$ of degree at most $n-2$, we have

$$\int_0^\infty P(u)F(u)\,du = 0;$$

Now, our goal is to show that F has at most $n-2$ positive real roots.

If so, we can choose $P(u)$ such that it changes sign exactly at these positive numbers

$\Rightarrow$ PF is either everywhere positive or everywhere negative;

$\Rightarrow$ $F \equiv 0$;

$\Rightarrow$ $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ coincide.

Sketch of the proof (main theorem) III

Claim: F has at most $n-2$ positive real roots.

Sketch of the proof (main theorem) III

Claim: F has at most $n-2$ positive real roots.

To find roots of $F(u) = 0$, we can solve

$$\left(1+\frac{u^2}{a_1^2}\right)\cdots\left(1+\frac{u^2}{a_n^2}\right) = \left(1+\frac{u^2}{b_1^2}\right)\cdots\left(1+\frac{u^2}{b_n^2}\right).$$

Sketch of the proof (main theorem) III

Claim: F has at most $n-2$ positive real roots.

To find roots of $F(u)=0$, we can solve

$$\left(1+\frac{u^2}{a_1^2}\right)\cdots\left(1+\frac{u^2}{a_n^2}\right)=\left(1+\frac{u^2}{b_1^2}\right)\cdots\left(1+\frac{u^2}{b_n^2}\right).$$

We can rewrite it as

$$1+u^2\underbrace{q(u^2)}_{\text{degree } n-2}+\frac{u^{2n}}{a_1^2\cdots a_n^2}=1+u^2\underbrace{r(u^2)}_{\text{degree } n-2}+\frac{u^{2n}}{b_1^2\cdots b_n^2},$$

and use equality of volumes to reduce it to

$$q(u^2)=r(u^2).$$

n-tuples of $\widetilde{V}_i$

Recall that

$$\widetilde{V}_i(K) = \frac{1}{n} \int\limits_{S^{n-1}} \rho_K^i(\theta) d\theta,$$

for any real i.

Thus,

$$\{\widetilde{V}_i\}_{i=1}^n \qquad \text{replace with} \qquad n\text{-tuple} \ \ \{\widetilde{V}_{i_k}\}_{k=1}^n$$

where $i_1, \ldots, i_n$ are distinct real numbers from the interval $(-2, n]$.

Theorem (Myroshnychenko, T., Yaskin (2020))

Let $\mathcal{E}_a$ and $\mathcal{E}_b$ be two ellipsoids in $\mathbb{R}^n$, centered at the origin such that $\widetilde{V}_{i_1}(\mathcal{E}_a) = \widetilde{V}_{i_1}(\mathcal{E}_b)$, $\widetilde{V}_{i_2}(\mathcal{E}_a) = \widetilde{V}_{i_2}(\mathcal{E}_b)$,..., $\widetilde{V}_{i_n}(\mathcal{E}_a) = \widetilde{V}_{i_n}(\mathcal{E}_b)$ for $i_k \in (-2, n] \setminus \{0\}$. Then $\mathcal{E}_a$ and $\mathcal{E}_b$ are congruent.

n-tuples of $\widetilde{V}_i$

We can proceed as before using that

$$\widetilde{V}_{i_k}(\mathcal{E}_a) = \widetilde{V}_{i_k}(\mathcal{E}_b) \quad \Rightarrow \quad \int_0^\infty u^{i_k - 1} F(u)\, du = 0, \text{ and}$$

Lemma (Eskenazis, Nayar, Tkocz (2018))

Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function that changes sign at most $N-1$ times in the interval $(0, \infty)$. If there exist N real numbers p_1,..., p_N such that

$$\int_0^\infty t^{p_k} f(t) dt = 0 \quad \text{for every} \quad k = 1, \ldots, N,$$

then f is identically equal to zero.

n-tuples of $\widetilde{V}_i$: remark

Theorem (Myroshnychenko, T., Yaskin (2020))

Let $\mathcal{E}_a$ and $\mathcal{E}_b$ be two ellipsoids in $\mathbb{R}^n$, centered at the origin such that $\widetilde{V}_{i_1}(\mathcal{E}_a) = \widetilde{V}_{i_1}(\mathcal{E}_b)$, $\widetilde{V}_{i_2}(\mathcal{E}_a) = \widetilde{V}_{i_2}(\mathcal{E}_b)$,..., $\widetilde{V}_{i_n}(\mathcal{E}_a) = \widetilde{V}_{i_n}(\mathcal{E}_b)$ for $i_k \in (-2, n]\setminus\{0\}$. Then $\mathcal{E}_a$ and $\mathcal{E}_b$ are congruent.

Remark: We can take larger intervals for i_k if we fix some conditions on dual volumes.

For example,

if $\widetilde{V}_n(\mathcal{E}_a) = \widetilde{V}_n(\mathcal{E}_b)$ and $\widetilde{V}_{i_k}(\mathcal{E}_a) = \widetilde{V}_{i_k}(\mathcal{E}_b)$ for $i_k \in (-2, n+2)\setminus\{0, n\}$, then conclusion is the same.

Intrinsic volumes in $\mathbb{R}^3$

Theorem (Petrov, Tarasov (2020))

Let $\mathcal{E}_a$ and $\mathcal{E}_b$ be two ellipsoids in $\mathbb{R}^3$ such that $V_1(\mathcal{E}_a) = V_1(\mathcal{E}_b)$, $V_2(\mathcal{E}_a) = V_2(\mathcal{E}_b)$, $V_3(\mathcal{E}_a) = V_3(\mathcal{E}_b)$. Then $\mathcal{E}_a$ and $\mathcal{E}_b$ are congruent.

Intrinsic volumes in $\mathbb{R}^3$

Theorem (Petrov, Tarasov (2020))

Let $\mathcal{E}_a$ and $\mathcal{E}_b$ be two ellipsoids in $\mathbb{R}^3$ such that $V_1(\mathcal{E}_a) = V_1(\mathcal{E}_b)$, $V_2(\mathcal{E}_a) = V_2(\mathcal{E}_b)$, $V_3(\mathcal{E}_a) = V_3(\mathcal{E}_b)$. Then $\mathcal{E}_a$ and $\mathcal{E}_b$ are congruent.

Alternative proof:

Using formulas which relate intrinsic (dual) volumes of ellipsoid and its polar

$$V_i(\mathcal{E}) = \frac{\kappa_i}{\kappa_n \kappa_{n-i}} V_n(\mathcal{E}) V_{n-i}(\mathcal{E}^\circ) \quad \text{and} \quad \widetilde{V}_i(\mathcal{E}) = \frac{1}{\kappa_n} \widetilde{V}_n(\mathcal{E}) \widetilde{V}_{n-i}(\mathcal{E}^\circ),$$

we can express

$$V_1(\mathcal{E}) = \frac{4}{V_3(\mathcal{E})} \widetilde{V}_4(\mathcal{E}) \quad \text{and} \quad V_2(\mathcal{E}) = \frac{9}{8\pi} V_3(\mathcal{E}) \widetilde{V}_{-1}(\mathcal{E}).$$

$$\{V_1(\mathcal{E}), V_2(\mathcal{E}), V_3(\mathcal{E})\} \longleftrightarrow \{\widetilde{V}_{-1}(\mathcal{E}), \widetilde{V}_3(\mathcal{E}) = V_3(\mathcal{E}), \widetilde{V}_4(\mathcal{E})\}.$$

Ellipsoids of revolution: part I

Theorem (Myroshnychenko, T., Yaskin (2020))

I. *Let* $\mathcal{E}_a$ *be* $\mathcal{E}_b$ *be two ellipsoids of revolution in* $\mathbb{R}^n$ *with semi-axes* $a_1, \dots, a_1, a_n$ *and* $b_1, \dots, b_1, b_n$ *such that* $V_n(\mathcal{E}_a) = V_n(\mathcal{E}_b)$, $V_i(\mathcal{E}_a) = V_i(\mathcal{E}_b)$, $V_{n-i}(\mathcal{E}_a) = V_{n-i}(\mathcal{E}_b)$ *for some* $i \neq n/2$, $1 \leq i \leq n-1$. *Then* $\mathcal{E}_a$ *and* $\mathcal{E}_b$ *are congruent.*

Ellipsoids of revolution: part I

Theorem (Myroshnychenko, T., Yaskin (2020))

I. *Let $\mathcal{E}_a$ be $\mathcal{E}_b$ be two ellipsoids of revolution in $\mathbb{R}^n$ with semi-axes $a_1, \dots, a_1, a_n$ and $b_1, \dots, b_1, b_n$ such that $V_n(\mathcal{E}_a) = V_n(\mathcal{E}_b)$, $V_i(\mathcal{E}_a) = V_i(\mathcal{E}_b)$, $V_{n-i}(\mathcal{E}_a) = V_{n-i}(\mathcal{E}_b)$ for some $i \neq n/2$, $1 \leq i \leq n-1$. Then $\mathcal{E}_a$ and $\mathcal{E}_b$ are congruent.*

Intrinsic volumes of ellipsoid of revolution can be represented as

$$V_i(\mathcal{E}_a) = C(n,i) a_1^i \int_{S^{n-1}} \left(\frac{a_n^2}{a_1^2} (\theta_1^2 + \cdots + \theta_i^2) + \theta_{i+1}^2 + \cdots + \theta_n^2 \right)^{1/2} d\theta,$$

where $C(n,i) = \dfrac{\binom{n}{i}}{n(2\kappa_n)^i \kappa_{n-i}}$.

Ellipsoids of revolution: part I

Theorem (Myroshnychenko, T., Yaskin (2020))

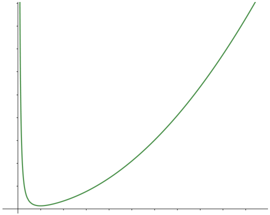
I. *Let $\mathcal{E}_a$ be $\mathcal{E}_b$ be two ellipsoids of revolution in $\mathbb{R}^n$ with semi-axes $a_1, \dots, a_1, a_n$ and $b_1, \dots, b_1, b_n$ such that $V_n(\mathcal{E}_a) = V_n(\mathcal{E}_b)$, $V_i(\mathcal{E}_a) = V_i(\mathcal{E}_b)$, $V_{n-i}(\mathcal{E}_a) = V_{n-i}(\mathcal{E}_b)$ for some $i \neq n/2$, $1 \leq i \leq n-1$. Then $\mathcal{E}_a$ and $\mathcal{E}_b$ are congruent.*

$$\begin{cases} V_n(\mathcal{E}_a) = V_n(B_2^n) \\ V_i(\mathcal{E}_a) = C(n,i) \int\limits_{S^{n-1}} \left(a_1^{2i-2n}(\theta_1^2 + \cdots + \theta_k^2) + a_1^{2i}(\theta_{i+1}^2 + \cdots + \theta_n^2) \right)^{1/2} d\theta. \end{cases}$$

not enough

$a_1 \to 0$ or $a_1 \to \infty$

$V_i(\mathcal{E}) \to \infty$



Sketch of the proof of part I

We can assume that $V_n(\mathcal{E}_a) = V_n(\mathcal{E}_b) = V_n(B_2^n)$. Then $\mathcal{E}_a$ and $\mathcal{E}_b$ have semi-axes $a_1, \dots, a_1^{-n+1}$ and $b_1, \dots, b_1^{-n+1}$. Goal: show that $a_1 = b_1$.

Sketch of the proof of part I

We can assume that $V_n(\mathcal{E}_a) = V_n(\mathcal{E}_b) = V_n(B_2^n)$. Then $\mathcal{E}_a$ and $\mathcal{E}_b$ have semi-axes $a_1, \dots, a_1^{-n+1}$ and $b_1, \dots, b_1^{-n+1}$. Goal: show that $a_1 = b_1$.

We can show that

$$V_i(\mathcal{E}_a) = V_i(\mathcal{E}_b) \quad \Longleftrightarrow \quad \widetilde{V}_{-1}(\mathcal{E}_a) = \widetilde{V}_{-1}(\mathcal{E}_b),$$

and

$$V_{n-i}(\mathcal{E}_a) = V_{n-i}(\mathcal{E}_b) \quad \underset{V_i(\mathcal{E})=\frac{\kappa_i}{\kappa_n \kappa_{n-i}} V_n(\mathcal{E}) V_{n-i}(\mathcal{E}^\circ)}{\Longleftrightarrow} \quad \widetilde{V}_{-1}(\mathcal{E}_a^\circ) = \widetilde{V}_{-1}(\mathcal{E}_b^\circ)$$

$$5 \quad \underset{\widetilde{V}_i(\mathcal{E})=\frac{1}{\kappa_n} \widetilde{V}_n(\mathcal{E}) \widetilde{V}_{n-i}(\mathcal{E}^\circ)}{\Longleftrightarrow} \quad \widetilde{V}_{n+1}(\mathcal{E}_a) = \widetilde{V}_{n+1}(\mathcal{E}_b).$$

Sketch of the proof of part I

$$\int_0^\infty u^{-2} F(u)\, du = 0 \qquad \text{and} \qquad \int_0^\infty u^{n} F(u)\, du = 0$$

where

$$F(u) = \frac{1}{(1 + a_1^{2i-2n} u^2)^{\frac{i}{2}} (1 + a_1^{2i} u^2)^{\frac{n-i}{2}}} - \frac{1}{(1 + b_1^{2i-2n} u^2)^{\frac{i}{2}} (1 + b_1^{2i} u^2)^{\frac{n-i}{2}}}.$$

Sketch of the proof of part I

$$\int_0^\infty u^{-2} F(u)\, du = 0 \qquad \text{and} \qquad \int_0^\infty u^n F(u)\, du = 0$$

where

$$F(u) = \frac{1}{(1 + a_1^{2i-2n} u^2)^{\frac{i}{2}} (1 + a_1^{2i} u^2)^{\frac{n-i}{2}}} - \frac{1}{(1 + b_1^{2i-2n} u^2)^{\frac{i}{2}} (1 + b_1^{2i} u^2)^{\frac{n-i}{2}}}.$$

Claim: $F(u)$ has at most one positive root.

Ellipsoids of revolution: part II

Theorem (Myroshnychenko, T., Yaskin (2020))

II. *If n is even, then V_n, $V_{n/2}$, V_i, for any i different from n and $n/2$, uniquely determine an ellipsoid of revolution (up to an isometry).*

Ellipsoids of revolution: part II

Theorem (Myroshnychenko, T., Yaskin (2020))

II. *If n is even, then V_n, $V_{n/2}$, V_i, for any i different from n and $n/2$, uniquely determine an ellipsoid of revolution (up to an isometry).*

Sketch of the proof:

Ellipsoids of revolution: part II

Theorem (Myroshnychenko, T., Yaskin (2020))

II. *If n is even, then V_n, $V_{n/2}$, V_i, for any i different from n and $n/2$, uniquely determine an ellipsoid of revolution (up to an isometry).*

Sketch of the proof:

- Show that

$$V_{n/2}(\mathcal{E}_a) = \text{constant} \cdot \int_0^\infty u^{-2} \left(1 - \frac{1}{\left((1+a_1^{-n}u^2)(1+a_1^n u^2)\right)^{\frac{n}{4}}}\right) du.$$

Ellipsoids of revolution: part II

Theorem (Myroshnychenko, T., Yaskin (2020))

II. *If n is even, then V_n, $V_{n/2}$, V_i, for any i different from n and $n/2$, uniquely determine an ellipsoid of revolution (up to an isometry).*

Sketch of the proof:

- Show that

$$V_{n/2}(\mathcal{E}_a) = \text{constant} \cdot \int_0^\infty u^{-2} \left(1 - \frac{1}{\left((1 + a_1^{-n} u^2)(1 + a_1^n u^2)\right)^{\frac{n}{4}}} \right) du.$$

- Note that $(1 + a_1^{-n} u^2)(1 + a_1^n u^2)$ is decreasing in a_1 on $(0, 1)$ and increasing in a_1 on $(1, \infty)$.

Ellipsoids of revolution: part II

Theorem (Myroshnychenko, T., Yaskin (2020))

II. *If n is even, then V_n, $V_{n/2}$, V_i, for any i different from n and $n/2$, uniquely determine an ellipsoid of revolution (up to an isometry).*

Sketch of the proof:

- Show that

$$V_{n/2}(\mathcal{E}_a) = \text{constant} \cdot \int_0^\infty u^{-2} \left(1 - \frac{1}{\left((1 + a_1^{-n} u^2)(1 + a_1^n u^2)\right)^{\frac{n}{4}}} \right) du.$$

- Note that $(1 + a_1^{-n} u^2)(1 + a_1^n u^2)$ is decreasing in a_1 on $(0, 1)$ and increasing in a_1 on $(1, \infty)$.
- $V_{n/2}(\mathcal{E}_a)$ is invariant under the transformation $a_1 \to \frac{1}{a_1}$.

Sketch of the proof of part II

$$\text{either} \quad \mathcal{E}_b = \mathcal{E}_a \quad \text{or} \quad \mathcal{E}_b = \mathcal{E}_a^\circ$$

Sketch of the proof of part II

$$\text{either} \quad \mathcal{E}_b = \mathcal{E}_a \quad \text{or} \quad \mathcal{E}_b = \mathcal{E}_a^\circ$$

if $\mathcal{E}_b = \mathcal{E}_a^\circ$, then $V_i(\mathcal{E}_a) = V_i(\mathcal{E}_a^\circ)$, which implies $V_{n-i}(\mathcal{E}_a) = V_{n-i}(\mathcal{E}_a^\circ)$.

Thus, by part I,

$$\mathcal{E}_a = \mathcal{E}_a^\circ \qquad \Rightarrow \qquad \mathcal{E}_a = B_2^n = \mathcal{E}_b.$$

Sketch of the proof of part II

$$\text{either} \quad \mathcal{E}_b = \mathcal{E}_a \quad \text{or} \quad \mathcal{E}_b = \mathcal{E}_a^\circ$$

if $\mathcal{E}_b = \mathcal{E}_a^\circ$, then $V_i(\mathcal{E}_a) = V_i(\mathcal{E}_a^\circ)$, which implies $V_{n-i}(\mathcal{E}_a) = V_{n-i}(\mathcal{E}_a^\circ)$.

Thus, by part I,

$$\mathcal{E}_a = \mathcal{E}_a^\circ \qquad \Rightarrow \qquad \mathcal{E}_a = B_2^n = \mathcal{E}_b.$$

Hence, $\mathcal{E}_a$ and $\mathcal{E}_b$ are the same.

Thank you!