A Combinatorial Perspective on Geometric Inequalities

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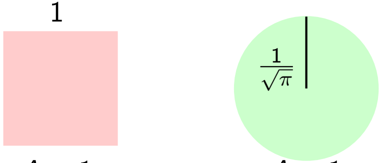
Geometric inequalities

Isoperimetric inequality in the plane

Of all planar regions of a given area, the disc has the smallest perimeter.

Related inequalities

Discrete Isoperimetric inequalities, Brunn-Minkowski, Prékopa-Leindler, Borell-Brascamp–Lieb, etc.



$A = 1$ $\qquad$ $A = 1$

$p = 4$ $\qquad$ $p = 2\sqrt{\pi} \approx 3.5$

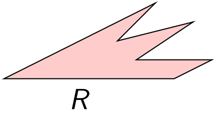
Stability of Geometric inequalities

Isoperimetric inequality in the plane

If R is a region with area π, then R has perimeter at least 2π. Equality happens if and only if R is a disc of radius 1.

Stability principle

If we are close to equality in isoperimetric inequality, then R is close to being a disc.



Stability of the isoperimetric inequality

Bonnesen, 1924

If R is a region with area π and perimeter at most $2\pi + \delta$, then R is sandwiched between two concentric discs with radii $1 - O(\sqrt{\delta})$ and $1 + O(\sqrt{\delta})$, respectively.

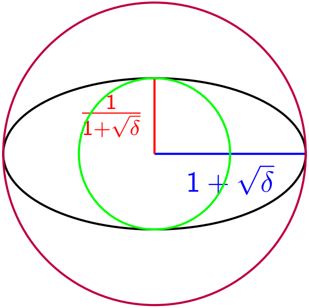
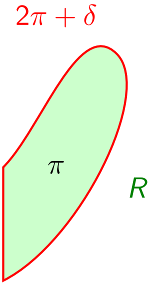


Figure: Ellipse with major and minor axes $1 + \sqrt{\delta}$ and $\frac{1}{1+\sqrt{\delta}}$. Area π and perimeter $2\pi + O(\delta)$. Inner and outer circles with radii $\approx 1 - \sqrt{\delta}$ and $1 + \sqrt{\delta}$.

Proof of stability

First step: find the center

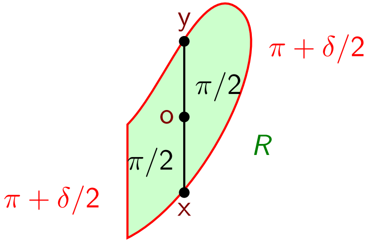
We want to show R is sandwiched between two discs of radii $1 \pm O(\sqrt{\delta})$.
Where is the center?



Proof of stability

First step: find the center

Find a line segment $\overline{xy}$ that divides both the perimeter and area in half and let o be the midpoint of $\overline{xy}$.

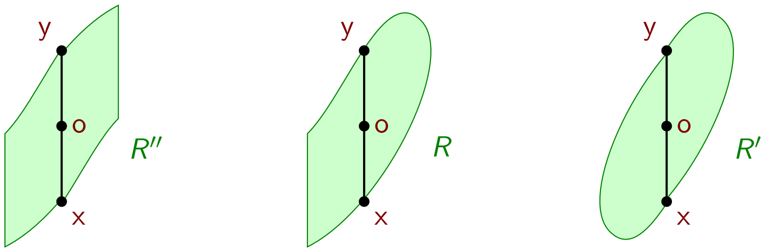


Proof of stability

Second step: reduce to the case when R is symmetric

From R construct two regions by erasing one half and reflecting the other half in o. Crucially, R, R' and R'' all have the area π and perimeter $2\pi + \delta$.

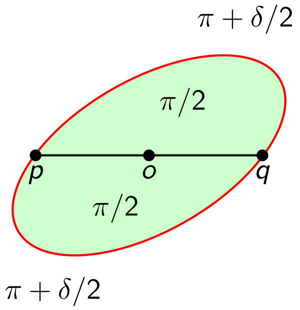
R is sandwiched between two discs centered at o if and only if R' and R'' are sandwiched between the same two discs.



Proof of stability

Third step: resolve the case when R is symmetric in o

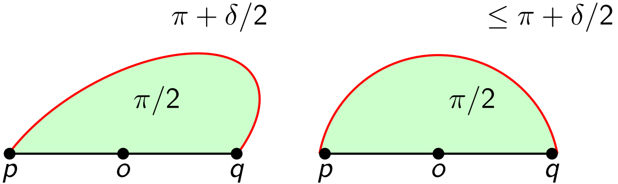
We assume R to be symmetric in o and show that R is sandwiched between two discs centered at the origin with radii $1 \pm O(\sqrt{\delta})$. This is equivalent to showing that for any segment $\overline{pq}$ through o, we have $2 + O(\sqrt{\delta}) \geq \overline{pq} \geq 2 - O(\sqrt{\delta})$.



Proof of stability

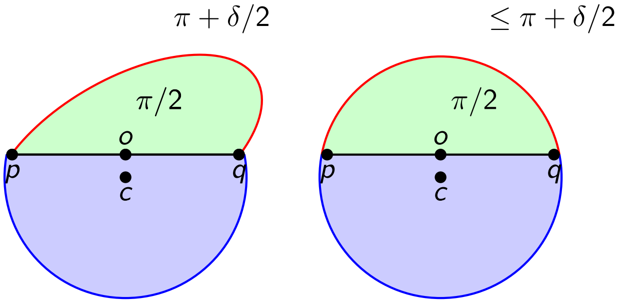
Third step: resolve the case when R is symmetric in o

The top half has area $\pi/2$ and red perimeter $\pi + \delta/2$. Consider a sector of a disc with chord $\overline{pq}$ that has area $\pi/2$. We claim that this has red perimeter at most $\pi + \delta/2$.



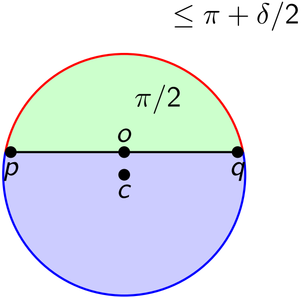
Third step: resolve the case when R is symmetric in o

Indeed, we can add the complementary sector of a disc to both figures and apply the isoperimetric inequality. Hence, the left figure has larger (red) perimeter than the right figure.



Third step: resolve the case when R is symmetric in o

A simple trigonometric computation in the disc allows us to express $\overline{pq}$ in terms of the area of the green sector and the red perimeter, giving the desired bound for $\overline{pq}$.



Sumsets

Minkowski sum

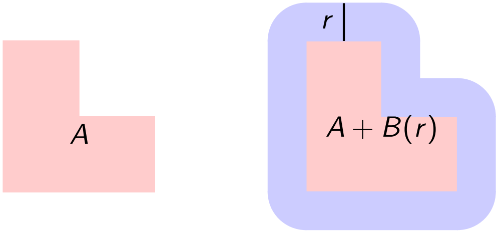
For $A, B \subset \mathbb{R}^d$,

$$A + B = \{x + y \colon x \in A,\ y \in B\}.$$

Example

If A is any set and B is a ball of radius r centered at origin, then,

$$A + B = \left\{z \in \mathbb{R}^d \colon \mathrm{dist}(z, A) \leq r\right\}.$$



Brunn-Minkowski inequality

Minkowski average

For $A, B \subset \mathbb{R}^d$,

$$\frac{A+B}{2} = \left\{ \frac{x+y}{2} : x \in A,\ y \in B \right\}.$$

Brunn 1887, Minkowski 1896

If $0 < t < 1$ and $A, B \subset \mathbb{R}^d$ have the same volume, then

$$|tA + (1-t)B| \geq |A|.$$

In particular,

$$\left| \frac{A+B}{2} \right| \geq |A|.$$

Equality in Brunn-Minkowski inequality

Brunn 1887, Minkowski 1896

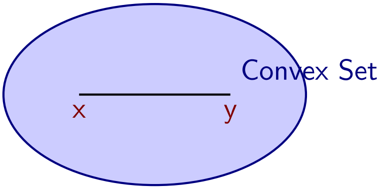
If $A, B \subset \mathbb{R}^d$ have the same volume, then

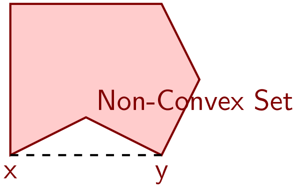
$$\left|\frac{A+B}{2}\right| \geq |A|.$$

Equality iff A and B are convex and equal up to translation.

Convex set

R is convex if for any points $x, y \in R$ the segment $\overline{xy}$ between them is contained in R.





Brunn-Minkowski inequality

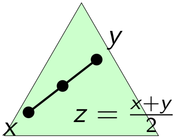
Brunn 1887, Minkowski 1896

If $A, B \subset \mathbb{R}^d$ have the same volume, then

$$\left|\frac{A+B}{2}\right| \geq |A|,$$

with equality iff A and B are convex and equal up to translation.

If $A = B$ is convex then $\frac{A+B}{2} = A$



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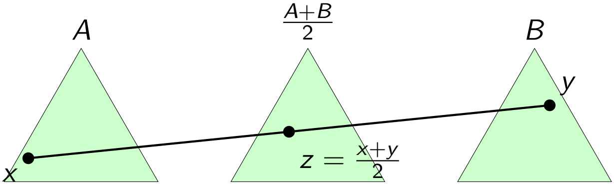

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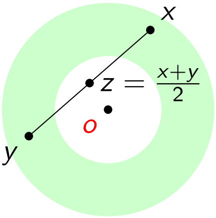
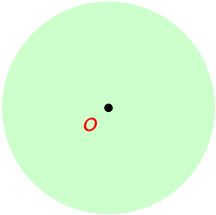
A=B is an annulus



$\frac{A+B}{2}$ is the outer disc



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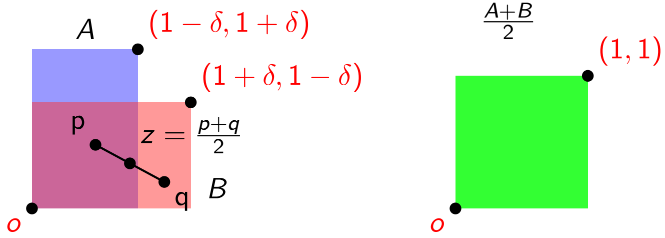

Figure: $A \neq B$ are convex. $|A| = |B| = 1 - \delta^2$; $|\frac{A+B}{2}| = 1$.

Stability of Brunn-Minkowski inequality

Brunn 1887, Minkowski 1896

If $A, B \subset \mathbb{R}^d$ have the same volume, then

$$\left|\frac{A+B}{2}\right| \geq |A|,$$

with equality iff A and B are convex and equal up to translation.

Stability principle

If we are close to equality, then A and B are close to being convex and equal up to translation.

Stability of Brunn-Minkowski inequality

First Folklore Conjecture

If $A, B \subset \mathbb{R}^d$ have the same volume and

$$\left|\frac{A+B}{2}\right| \leq (1+\delta)|A|, \text{ where } \delta \ll 1,$$

then, up to translation, $|A \triangle B| \leq O(\sqrt{\delta})|A|$.

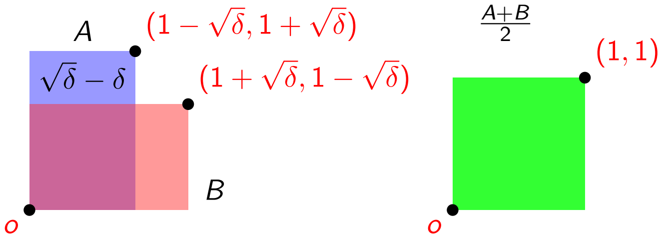

Figure: $|A| = |B| = 1-\delta$, $|\frac{A+B}{2}| = 1$; $|A \triangle B| = 2\sqrt{\delta} - 2\delta$.

Stability of Brunn-Minkowski inequality

Second Folklore conjecture

If $A, B \subset \mathbb{R}^d$ have the same volume and $|\frac{A+B}{2}| \leq (1+\delta)|A|$, where $\delta \ll 1$, then $|\text{co}(A) \setminus A|, |\text{co}(B) \setminus B| \leq O(\delta)|A|$.

$\text{co}(X)$ is the smallest convex set containing X.

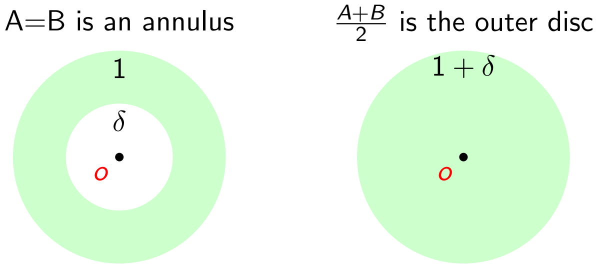

Figure: $|A| = |B| = 1$, $|\frac{A+B}{2}| = 1+\delta$, $|\text{co}(A) \setminus A| = \delta$ where $\text{co}(A)$ is outer disc.

When one of the sets is convex

Folklore conjectures

If $A, B \subset \mathbb{R}^d$ have the same volume and $|\frac{A+B}{2}| \leq (1+\delta)|A|$, then, up to translation, $|A \triangle B| \leq O(\sqrt{\delta})|A|$. Also, $|\text{co}(A) \setminus A|, |\text{co}(B) \setminus B| \leq O(\delta)|A|$.

Figalli, Maggi, Pratelli 2009

Resolved the first conjecture when A and B are convex.

When one of the sets is convex

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Resolved the first conjecture when A is a ball and B is arbitrary.

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Barchiesi, Julin 2017

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When both sets are arbitrary

Folklore conjectures

If $A, B \subset \mathbb{R}^d$ have the same volume and $|\frac{A+B}{2}| \leq (1+\delta)|A|$, then, up to translation, $|A \triangle B| \leq O(\sqrt{\delta})|A|$ and $|\mathrm{co}(A) \setminus A|, |\mathrm{co}(B) \setminus B| \leq O(\delta)|A|$.

Figalli, Jerison 2014

Established sub-optimal bounds for both conjectures of the form

$$|A \triangle B|, |\mathrm{co}(A) \setminus A| \leq \delta^{\exp^{-\exp(d)}}|A|.$$

Results

Folklore conjectures

If $A, B \subset \mathbb{R}^d$ have the same volume and $|\frac{A+B}{2}| \leq (1+\delta)|A|$, then, up to translation, $|A \triangle B| \leq O(\sqrt{\delta})|A|$. Also, $|\text{co}(A) \setminus A|, |\text{co}(B) \setminus B| \leq O(\delta)|A|$.

van Hintum, Spink, Tiba 2019

Resolved both conjectures in the plane.

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Figalli, van Hintum, Tiba 2023

Resolved both conjectures in all dimensions.

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van Hintum, Spink, Tiba 2019

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Figalli, van Hintum, Tiba 2023

Resolved both conjectures in all dimensions.

van Hintum, Spink, Tiba 2019

Determined the optimal constant when $A = B$ in dimension ≤ 4 and the asymptotic constant in all dimensions.

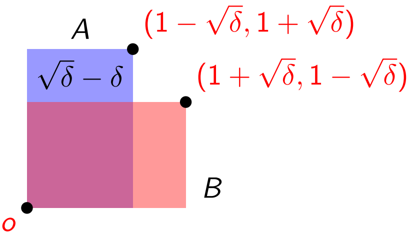
Results

Theorem Figalli, van Hintum, Tiba (2023)

If $A, B \subset \mathbb{R}^d$ have the same volume and

$$|tA + (1-t)B| \leq (1+\delta)|A|, \text{ where } \delta \ll_{d,t} 1,$$

then, up to translation, $|A \triangle B| \leq O_d(\sqrt{\delta/t})|A|$.



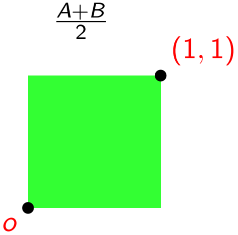


Figure: $|A| = |B| = 1-\delta$, $|\frac{A+B}{2}| = 1$; $|A \triangle B| = 2\sqrt{\delta} - 2\delta$.

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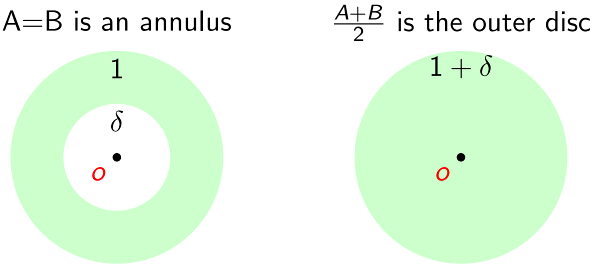


Figure: $|A| = |B| = 1$, $|\frac{A+B}{2}| = 1+\delta$, $|\mathrm{co}(A) \setminus A| = \delta$ where $\mathrm{co}(A)$ is outer disc.

Brunn 1887, Minkowski 1896

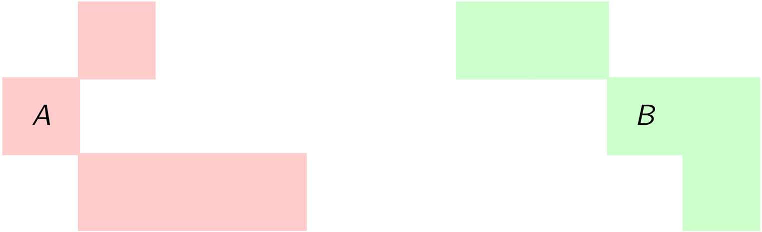
If $A, B \subset \mathbb{R}^d$ have the same volume then $|(A+B)/2| \geq |A|$.

Proof

1. Do parallel hyperplane cuts to partition $A = \sqcup A_i$ and $B = \sqcup B_i$ s.t. $|A_i| = |B_i|$ and $(A_i + B_i)/2$ are disjoint.

2. Prove BM inequality for A_i and B_i i.e. $|(A_i + B_i)/2| \geq |A_i|$.

Conclude $|(A+B)/2| \geq \sum_i |(A_i + B_i)/2| \geq \sum_i |A_i| = |A|$.



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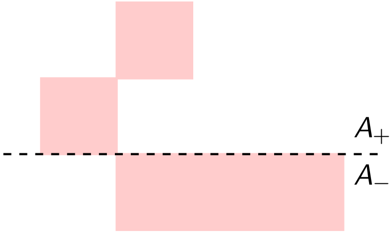

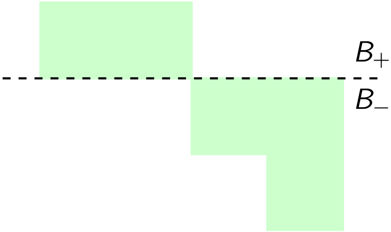

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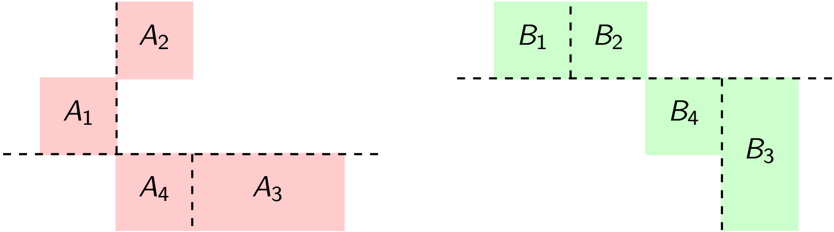

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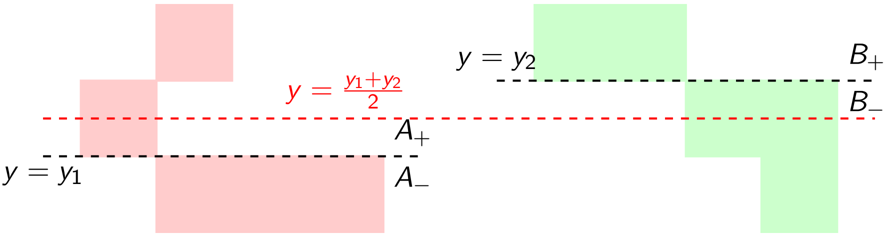

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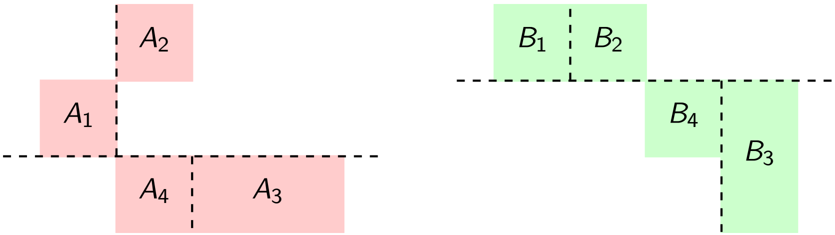

Theorem (Figalli, van Hintum, Tiba)

If $A, B \subset \mathbb{R}^d$ have the same volume and $|\frac{A+B}{2}| \leq (1+\delta)|A|$ where $\delta \ll 1$ then, up to translation, $|A \triangle B| \leq O(\sqrt{\delta})|A|$.

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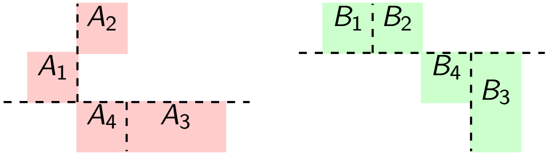
Claim: $(1+\delta)|A| \geq |(A+B)/2| \geq \sum_i |(A_i+B_i)/2| = \sum_i (1+\delta_i)|A_i|$.

2. Prove BM stability for A_i and B_i: $\exists\ z$ s.t. $|A_i \triangle (z + B_i)| \leq O(\sqrt{\delta_i})|A_i|$.

Conclude $|A \triangle (z+B)| \leq \sum_i |A_i \triangle (z+B_i)| \leq \sum_i O(\sqrt{\delta_i})|A_i| \leq O(\sqrt{\delta})|A|$.



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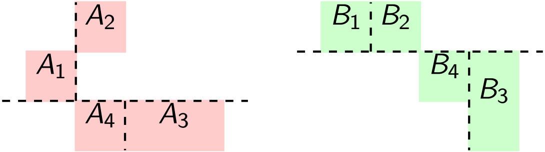
1. Do parallel hyperplane cuts to partition $A = \sqcup A_i$ and $B = \sqcup B_i$ as before $|A_i| = |B_i|$, $(A_i + B_i)/2$ disjoint. Say $|(A_i + B_i)/2| = (1 + \delta_i)|A_i|$.

Claim: $(1+\delta)|A| \geq |(A+B)/2| \geq \sum_i |(A_i + B_i)/2| = \sum_i (1+\delta_i)|A_i|$.

2. Prove BM stability for A_i and B_i: $\exists\ z_i$ s.t. $|A_i \triangle (z_i + B_i)| \leq O(\sqrt{\delta_i})|A_i|$

! The translates z_i are not the same !

Conclude $|A \triangle (z+B)| \leq \sum_i |A_i \triangle (z + B_i)| \leq \sum_i O(\sqrt{\delta_i})|A_i| \leq O(\sqrt{\delta})|A|$.



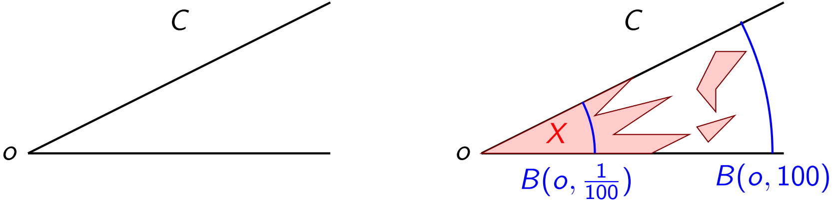
Cone like sets

Cone

$C \in \mathbb{R}^d$ is a cone with vertex at origin o if $C = H_1^+ \cap \cdots \cap H_n^+$, where $H_1, \ldots, H_n$ are hyperplanes passing through the origin o.

Cone-like set

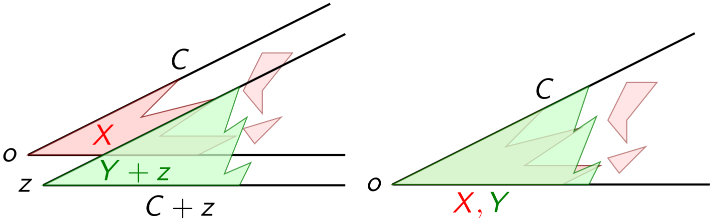
$X \subset C$ is 100-C-like if $C \cap B(o, 1/100) \subset X \subset C \cap B(o, 100)$



Cone like sets

Lemma

Say C is a cone and $X, Y \subset C$ are 100-C-like sets. Assume that $|X| = |Y|$, $|(X+Y)/2| \leq (1+\delta)|X|$ and $\exists z$ s.t $|X \triangle (Y+z)| = O(\sqrt{\delta})|X$
Then, $|X \triangle Y| = O(\sqrt{\delta})|X|$ i.e. up to constants the optimal translate is 0.



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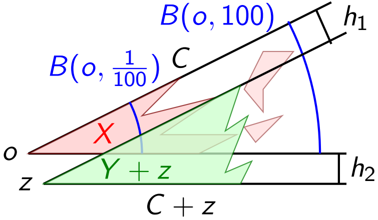
Proof. Affine transform so $\angle\alpha = 30^\circ$, which implies $|X| = |Y| = c$.

Claim $|z| \leq c\sqrt{\delta}$. Enough $h_i \leq c\sqrt{\delta}$. Note $R \subset X \triangle (Y+z)$ so $|R| \leq c\sqrt{\delta}$, but $|R| \geq ch_1$. Dream $|X \triangle Y| = |X \triangle (Y+z)| + c|z| \leq c\sqrt{\delta}$.
True if X, Y are (nearly) convex. Other Main Thm $|\text{co}(X) \setminus X| \leq O(\delta)|X|$



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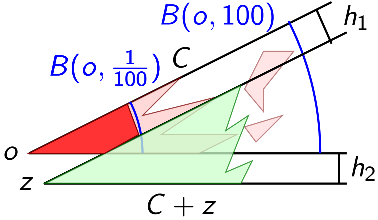
$R = C \cap B(o, \frac{1}{100}) \setminus (C+z)$
$R \subset X \triangle (Y+z)$



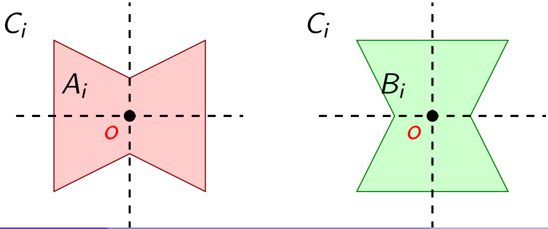
Theorem (Figalli, van Hintum, Tiba)

If $A, B \subset \mathbb{R}^d$ have the same volume and $|\frac{A+B}{2}| \leq (1+\delta)|A|$ where $\delta \ll 1$ then, up to translation, $|A \triangle B| \leq O(\sqrt{\delta})|A|$.

Proof Revised

1. Do hyperplane cuts to partition $\mathbb{R}^d = \sqcup C_i$, where C_i are arbitrary narrow cones at origin s.t. 1. $A_i, B_i \subset C_i$ are 100-C_i-like and 2. $|A_i| = |B_i|$

2. Prove BM stability for A_i and B_i: $\exists\, z_i$ s.t. $|A_i \triangle (z_i + B_i)| \leq O(\sqrt{\delta_i})|A_i|$
! Optimal translates $z_i = 0$ coincide !
Conclude $|A \triangle (z+B)| \leq \sum_i |A_i \triangle (z + B_i)| \leq \sum_i O(\sqrt{\delta_i})|A_i| \leq O(\sqrt{\delta})|A|$.



Theorem (Figalli, van Hintum, Tiba)

If A and B have the same volume and $|\frac{A+B}{2}| \leq (1+\delta)|A|$ where $\delta \ll 1$, then, up to translation, $|A \triangle B| \leq O(\sqrt{\delta})|A|$.

Proof Revised

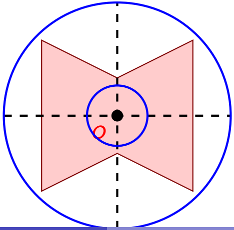
1. Do hyperplane cuts to partition $\mathbb{R}^d = \sqcup C_i$, where C_i are arbitrary narrow cones at origin s.t. 1. $A_i, B_i \subset C_i$ are 100-C_i-like and 2. $|A_i| = |B_i|$

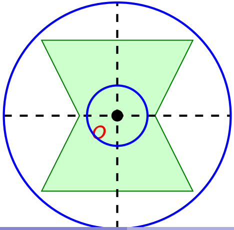
! 1 is always satisfied; 2 is interesting !

2. Prove BM stability for A_i and B_i: $\exists\, z$ s.t. $|A_i \triangle (z + B_i)| \leq O(\sqrt{\delta_i})|A_i|$.

! Optimal translates $z_i = 0$ coincide !

Conclude $|A \triangle (z + B)| \leq \sum_i |A_i \triangle (z + B_i)| \leq \sum_i O(\sqrt{\delta_i})|A_i| \leq O(\sqrt{\delta})|A|$.

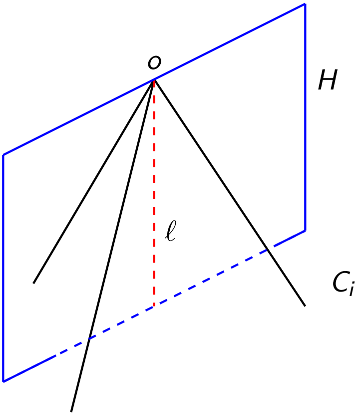




Refining move in $\mathbb{R}^3$

Lemma

Let C_i be a cone such that inside C_i we have $|A_i| = |B_i|$. Let ℓ be a line through the origin o. There exists a plane H through ℓ which partitions $C_i = C_i^+ \sqcup C_i^-$ such that $|A_i^+| = |B_i^+|$ and $|A_i^-| = |B_i^-|$.



Refining the partition of $\mathbb{R}^3$ into narrow cones

Game

At each stage we choose a cone C_i, we choose a line ℓ through o and then the enemy chooses a plane H through ℓ dividing the cone C_i into two smaller cones.

Hope

Can we play the game to produce a partition into arbitrarily narrow cones?

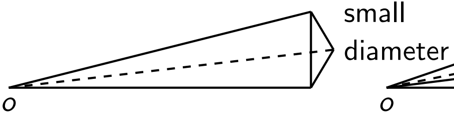
Refining the partition of $\mathbb{R}^3$ into narrow cones

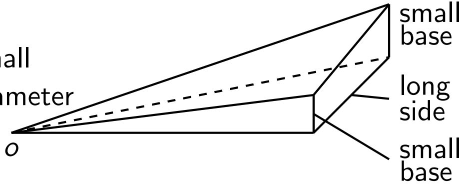
Theorem (Figalli, van Hintum, Tiba)

We can play the game to produce a partition $\mathbb{R}^3 = C_1 \sqcup \cdots \sqcup C_n$ where each cone C_i falls into one of two categories:

1. C_i has $O(1)$ faces and is arbitrarily narrow.

2. C_i is trapezoidal and is arbitrarily narrow in the direction of the base.





Refining the partition of $\mathbb{R}^3$ into narrow cones

Theorem (Figalli, van Hintum, Tiba)

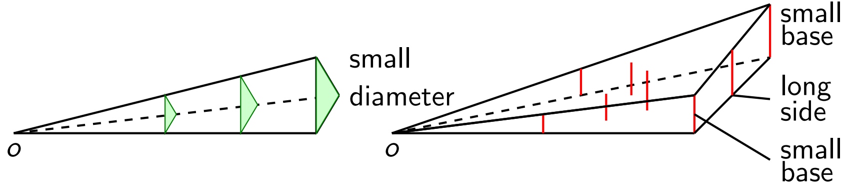
In both cases, the sets A_i (and B_i) inside C_i are *simple*:

1.Every section parallel to a given plane is entirely in A_i or disjoint from A_i

2.Every fiber parallel to the basis is entirely in A_i or disjoint from A_i

Theorem (Figalli, van Hintum, Tiba)

For simple sets A_i and B_i with the same volume, if $|\frac{A_i+B_i}{2}| \leq (1+\delta)|A_i|$ where $\delta \ll 1$, then, up to translation, $|A_i \triangle B_i| \leq O(\sqrt{\delta})|A_i|$.



First Main Result

Theorem (Figalli, van Hintum, Tiba)

If $A, B \subset \mathbb{R}^d$ have the same volume and

$$|tA + (1-t)B| \leq (1+\delta)|A|, \text{ where } \delta \ll_{d,t} 1,$$

then, up to translation, $|A \triangle B| \leq O_d(\sqrt{\delta/t})|A|$.

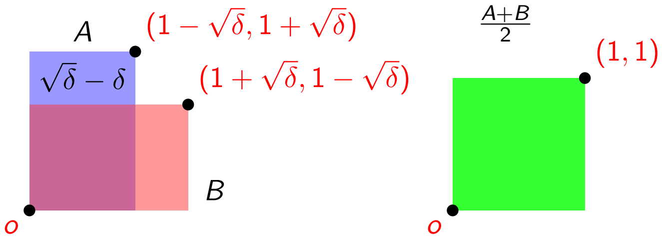


Figure: $|A| = |B| = 1-\delta$, $|\frac{A+B}{2}| = 1$; $|A \triangle B| = 2\sqrt{\delta} - 2\delta$.

Second Main Result

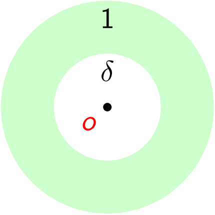
Theorem (Figalli, van Hintum, Tiba)

If $A, B \subset \mathbb{R}^d$ have the same volume and

$$|tA + (1-t)B| \leq (1+\delta)|A|, \text{ where } \delta \ll_{d,t} 1,$$

then $|co(A) \setminus A|, |co(B) \setminus B| \leq O_{d,t}(\delta)|A|$.
$co(X)$ is the smallest convex set containing X

A=B is an annulus



$\frac{A+B}{2}$ is the outer disc

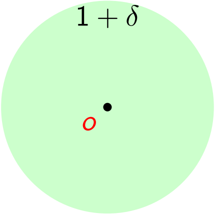


Figure: $|A| = |B| = 1$, $|\frac{A+B}{2}| = 1 + \delta$, $|co(A) \setminus A| = \delta$ where $co(A)$ is outer disc.

Optimal dependency on t in linear BM stability

Conjecture

If $A, B \subset \mathbb{R}^d$ have the same volume and

$$|tA + (1-t)B| \leq (1+\delta)|A|, \text{ where } \delta \ll_{d,t} 1,$$

then

$$|\mathrm{co}(A) \setminus A| \leq O_d(t^{-1}\delta)|A| \text{ and } |\mathrm{co}(B) \setminus B| \leq O_d(t^{-d+1}\delta)|A|.$$

Higher values of δ in linear BM stability

Conjecture

If $A \subset \mathbb{R}^d$ and

$$\left|\frac{A+A}{2}\right| \le (1+\delta)|A|, \text{ where } \delta \ll_{d,t} 1,$$

then $|\mathrm{co}(A) \setminus A| \le (\frac{2^d}{d} + o(1))\delta|A|$.

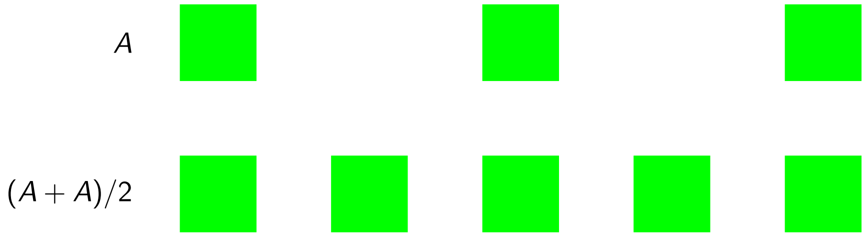
Higher values of δ in linear BM stability

Conjecture

If $A \subset \mathbb{R}^d$ and

$$\left|\frac{A+A}{2}\right| \leq 1.99|A|,$$

then there is a convex set K with $|K| = |A|$ such that $|K \cap A| \geq \Omega(1)|A|$.



Stability of Prékopa-Leindler

Prékopa-Leindler

Let $f, g\colon \mathbb{R}^d \to \mathbb{R}_+$ be continuous with bounded support and $\int f = \int g = 1$. Define $h(z) = \sup_{z=\frac{x+y}{2}} \sqrt{f(x)g(y)}$. Then $\int h \geq 1$.

Equality

Equality holds if and only if there exists $a \in \mathbb{R}^d$ such that $f(x) = g(x+a)$ is log-concave i.e. $f(tx + (1-t)y) \geq f^t(x)f^{1-t}(y)\ \forall t \in (0,1), x, y \in \mathbb{R}^d$.

Conjecture (Borőczky, Figalli and Ramos)

If $\int h \leq 1 + \delta$, then, up to replacing $g(x) := g(x+a)$ for some $a \in \mathbb{R}^d$, there exists a log-concave function $\ell\colon \mathbb{R}^d \to \mathbb{R}_+$ such that $\int |f - \ell| + |g - \ell| \leq O_d(\sqrt{\delta})$.

Discrete setting higher dimensions

Degenerate sets

Sets in $\mathbb{Z}^d$ can look like sets in $\mathbb{Z}$ e.g. the set $I = \{(0,0), (1,0), \ldots, (n,0)\}$ has $I + I = \{(0,0), \ldots, (2n,0)\}$ so $|I + I| = 2|I| - 1$.

Green-Tao theorem

Given $d \in \mathbb{N}, \epsilon > 0$ there exists $n \in \mathbb{N}$ such that if $A \subset \mathbb{Z}^d$ is not covered by n parallel hyperplanes, then $|A + A| \geq (2^d - \epsilon)|A|$

van Hintum, Spink, Tiba 2020

If $d \in \mathbb{N}, \delta > 0$ there exists $n \in \mathbb{N}$ such that if $A \subset \mathbb{Z}^d$ is not covered by n parallel hyperplanes and if $|A + A| \leq (2^d + \delta)|A|$, then A is contained inside a convex progression P i.e. convex set intersected a sub-lattice of $\mathbb{Z}^d$ with size $|P| \leq (1 + O(\delta))|A|$.

Discrete setting higher dimensions

van Hintum, Keevash, Tiba 2023

Given $d \in \mathbb{N}, \epsilon > 0$ there exists $n \in \mathbb{N}$ such that if $A, B \subset \mathbb{Z}^d$ have the same size and B is not covered by n parallel hyperplanes, then $|A + B| \geq (2^d - \varepsilon)|A|$. $n = O_d(\varepsilon^{-1})$ is optimal.

Campos, van Hintum, Keevash, Tiba 2023

If $d \in \mathbb{N}, \delta > 0$ there exists $n \in \mathbb{N}$ such that the following holds. Assume $A, B \subset \mathbb{Z}^d$ have the same size, are not covered by n parallel hyperplanes and $|A + B| \leq (2^d + \delta)|A|$. Then, up to translation, both A and B are contained inside a convex progression P i.e. convex set intersected a sub-lattice of $\mathbb{Z}^d$ with size $|P| \leq (1 + O(\sqrt{\delta}))|A|$.