Optimal transport maps, majorization, and log-subharmonic measures

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Joint work with Guido De Philippis

Regularity of transport maps

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Transport maps. A map $T : \mathbb{R}^n \to \mathbb{R}^n$ **transports** a source probability measure μ on $\mathbb{R}^n$ to a target probability measure ν on $\mathbb{R}^n$ if, for all continuous and bounded $\eta : \mathbb{R}^n \to \mathbb{R}$,

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Regularity. We are interested in controlling the eigenvalues of the derivative $\mathrm{D}T$ of T uniformly in $x \in \mathbb{R}^n$, e.g.,

$$|\mathrm{D}T(x)|_{\mathrm{op}}, \qquad \mathrm{Tr}[\mathrm{D}T(x)], \qquad \det[\mathrm{D}T(x)].$$

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$$\begin{aligned}
&\mathrm{Var}_\nu[\eta] = \mathrm{Var}_\mu[\eta \circ T] \leq c_\mu \int |\nabla(\eta \circ T)|^2 \,\mathrm{d}\mu \\
&\leq c_\mu \, \|\mathrm{D}T\|_{L^\infty}^2 \int |(\nabla \eta) \circ T|^2 \,\mathrm{d}\mu = c_\mu \, \|\mathrm{D}T\|_{L^\infty}^2 \int |\nabla \eta|^2 \,\mathrm{d}\nu \\
&\leq \ell^2 c_\mu \int |\nabla \eta|^2 \,\mathrm{d}\nu.
\end{aligned}$$

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Note. Volume contraction is significantly weaker than Lipschitz regularity since only the product of the eigenvalues of $\mathrm{D}T$ is controlled rather than the individual eigenvalues.

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Proof. By the change of variables formula,

$$f(x)\gamma(x) = \gamma(T(x))\underbrace{|\det \mathrm{D}T(x)|}_{\leq 1} \leq \gamma(T(x)) \leq \frac{1}{(2\pi)^{\frac{n}{2}}}.$$

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$$\int_{\mathbb{R}^n} \varphi(\mu(x)) \, \mathrm{d}x \leq \int_{\mathbb{R}^n} \varphi(\nu(x)) \, \mathrm{d}x \qquad \forall \text{ convex } \varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}.$$

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Taylor expansion of φ, and the fact $\int \mu = \int \nu$, imply that it suffices to show for each $r \geq 0$,

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Using $[sx - r]^+ \le s[x - r]^+$ for $s \in (0,1)$ (since $x \mapsto [x - r]^+$ is convex vanishing at 0),

$$\begin{aligned} \int [\mu(x) - r]^+ \, \mathrm{d}x &= \int [\nu(T(x)) |\det \mathrm{D}T(x)| - r]^+ \, \mathrm{d}x \\ &\le \int |\det \mathrm{D}T(x)| [\nu(T(x)) - r]^+ \, \mathrm{d}x = \int [\nu(x) - r]^+ \, \mathrm{d}x. \end{aligned}$$

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Example. ν has higher q-Rényi entropy than μ.

Coherent states

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Coherent states are states which are as classical as possible without violating the uncertainty principle.

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A **Glauber state**

$$\psi_{q,p}(x) := e^{\frac{i}{\hbar}\langle x,p\rangle}\left[(2\pi\hbar)^{-d}\exp\left(-\frac{|x-q|^2}{2\hbar}\right)\right]$$

increases the uncertainty around a position q with a Gaussian window of variance $\hbar$, which is the minimum allowed by the uncertainty principle.

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$$\mathcal{L}\psi(q,p) := e^{\frac{i}{2}\langle p,q\rangle} \int_{\mathbb{R}^d} e^{i\langle y,p\rangle} e^{-\frac{|y-q|^2}{2}} \psi(y)\,\mathrm{d}y.$$

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Similarly for **mixed states**, $\sum_{j=1}^k a_j\psi_j$ with $a_j \geq 0$, $\sum_{j=1}^k a_j = 1$, and $\{\psi_j\}_{j=1}^k$ orthonormal, $(2\pi\hbar)^{-d}\sum_{j=1}^k a_j|\mathcal{L}\psi_j|^2$ is the corresponding probability measure over phase space.

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Wehrl conjecture. For each mixed state,

$$\begin{aligned}\text{Entropy}\left[|\mathcal{L}(\text{mixed state})|^2\right] &\geq \text{Entropy}\left[|\mathcal{L}(\text{Glauber state})|^2\right] \\ &= \text{Entropy}\left[\text{Gaussian}\right].\end{aligned}$$

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Proofs. The Wehrl conjecture was first proven by Lieb, and since then the generalized Wehrl conjecture was proven also for other groups, but some conjectures remain open.

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Thus, the **main question** is under what conditions on μ and ν do we have a volume-contracting map T between μ and ν?

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Remark 1. Theorem is sharp: $\mu = \mathcal{N}(0, \sigma^2 \mathrm{Id}_n)$ and $\nu = \mathcal{N}(0, \mathrm{Id}_n)$ imply $\nabla\Phi(x) = \frac{x}{\sigma}$ so $\Delta\Phi(x) = \frac{n}{\sigma} = n\sqrt{\frac{\alpha_{\mathsf{up}}}{\alpha_{\mathsf{low}}}}$.

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Remark 2. Theorem implies the Lipschitz bound $\|\nabla^2 \Phi\|_{L^\infty} \leq n \sqrt{\frac{\alpha_{\mathsf{up}}}{\alpha_{\mathsf{low}}}}$ which is sharp in the limit $\epsilon \to 0$,

$$\mu = \mathcal{N}(0, \Sigma_\epsilon), \quad \mu = \mathcal{N}(0, \mathrm{Id}_n), \quad \Sigma_\epsilon = \mathrm{diag}\left(\frac{1}{n}, \frac{1}{\epsilon}, \ldots, \frac{1}{\epsilon}\right).$$

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Remark 3. By AM-GM inequality $\nabla \Phi$ is volume-contracting,

$$\|\det \nabla^2 \Phi\|_{L^\infty} \leq \left(\frac{\alpha_{\text{up}}}{\alpha_{\text{low}}}\right)^{\frac{n}{2}}.$$

Log-subharmonic measures

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Thus,

$$f(x) \leq e^{\frac{|x|^2}{2}} \quad \text{and} \quad \gamma \text{ majorizes } f\gamma.$$

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- $f(z) = \exp\left(-\sum_{i<j=1}^{N} \log|z_i - z_j|\right)$ for $z \in \mathbb{C}$.

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  In particular we get majorization for **two-dimensional Coulomb gases**.

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  This gives a new proof for the generalized Wehrl conjecture for Glauber states since for each wave function ψ,

  $$|\mathcal{L}\psi(q,p)|^2 = 2^{-\frac{d}{2}} |\tilde{f}(q+ip)|^2 e^{-\pi(|q|^2+|p|^2)}$$

  where $\tilde{f}$ is entire.

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In particular this gives stability in the (original) Wehrl conjecture for Glauber states.

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$$[0,1] \ni t \mapsto \int_{\mathbb{R}^n} \varphi(\rho_t(x))\,\mathrm{d}x \text{ is monotonically non-decreasing.}$$

Proof of $\Delta V \le \alpha_{\mathsf{up}} n$ and $\nabla^2 W \succeq \alpha_{\mathsf{low}} \operatorname{Id}_n \implies \|\Delta\Phi\|_{L^\infty} \le n\sqrt{\frac{\alpha_{\mathsf{up}}}{\alpha_{\mathsf{low}}}}$

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Making this argument rigorous is quite technical and relies on Kolesnikov's L^p method as well as approximation arguments.

The (inverse) Kim-Milman map

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Given $\mu = f\nu$ with $\nu = \gamma = \mathcal{N}(0, \mathrm{Id}_n)$ let

$$\partial_t S_t(x) = -\nabla \log P_t f(S_t(x)), \qquad S_0(x) = x,$$

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The **(inverse) Kim-Milman map** is $T_{\mathrm{km}} := S_\infty$ which transports μ to γ.

Lipschitz regularity for the Kim-Milman map

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- Nowadays there are many more Lipschitz regularity results for the Kim-Milman map and its reverse than for the Brenier map.
- In particular, Mikulincer-S. showed that for $\mathrm{d}\mu = e^{-V}\,\mathrm{d}x$ we have

$$\nabla^2 V \le \alpha_{\mathsf{up}} \mathrm{Id}_n \quad \Longrightarrow \quad \|\mathrm{D}\, T_{\mathsf{km}}\|_{L^\infty} \le \sqrt{\alpha_{\mathsf{up}}}.$$

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Q. Can we get the analogue of the De Philippis, S, result

$\Delta V \leq \alpha_{\mathsf{up}} n \Longrightarrow \| \operatorname{Tr}[\mathrm{D}\, T_{\mathsf{km}}]\|_{L^\infty} \leq n\sqrt{\alpha_{\mathsf{up}}}$ and $\| \det \mathrm{D}\, T_{\mathsf{km}}\|_{L^\infty} \leq \alpha_{\mathsf{up}}^{\frac{n}{2}}$?

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We cannot establish the trace bound but can “show” that

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To quotation marks are because that establishing the existence of $T_{\mathsf{km}} = S_\infty$ under low regularity is not clear.

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Show that $\Delta \log P_t f$ can be controlled under log-subharmonicity results on f.

Use Grönwall's inequality to bound $\det \mathrm{D}S_t$ and take $t \to \infty$.

Thank You