Consider a product $\Sigma \times S^1$ with a Riemannian metric without conjugate points, where Σ is a surface of higher genus. What can one say about this metric? Of course, it does not have to be a product, there maybe a twist, however the cover with respect to the S^1 factor possibly has to be a product. This is some sort of a generalized Hopf Conjecture, though Σ carries a lot of negative curvature.

Consider a compact surface $\Sigma^m \subset M^n$ with a boundary $\partial\Sigma^m$. Assume that the surface is totally convex in the following sense: it intrinsically has no conjugate points and every shortest path between any two points of Σ entirely belongs to Σ. Is this true that Σ is area minimizing, that is: for any Σ_1 such that $\partial\Sigma_1 = \partial\Sigma$, $Area_m\Sigma_1 \geq Area_m\Sigma$? (Note the difference with minimal surfaces, for which just the differential of the surface area functional vanishes). We also assume that intrinsically $Sigma$ has no conjugate points.

This is true for $n = m + 1$. If Σ is *stably* minimizing, that is $k\Sigma$ (with the boundary $k\partial\Sigma$) is also minimizing for all k, there is a hope to prove the statement by constructing a calibrating form (a closed m-form ω such that $||\omega|| \leq 1$ and ω coincides with the area form on the surface Σ). However, if Σ is not stably minimizing, such a form does not exist, and I see no approaches so far. Furthermore, the statement may even be false. There are counter-examples (due to S. Ivanov and me) in normed spaces with respect to the symplectic (Holmes-Thompson) surface area and Σ being a chain over $\mathbb{Z}$.

We say that a metric graph is uniformly bounded if the degrees of all vertices are uniformly bounded and the lengths of edges are pinched between two positive constants; a metric space is approximable by a uniform graph if there is one within a finite Gromov-Hausdorff distance from the space. Is $\mathbb{R}^3$ approximable? $\mathbb{R}^2$ is. Is there a Riemannian manifold with bounded geometry which it is not approximable. Gromov hyperbolic geodesic spaces with bounded geometry are approximable.

Consider a 4-dim Riemannian manifold M^4 with $K_\sigma \geq 0$ and a 2-torus $\mathbb{T}^2 \subset M^4$. Assume that the torus it totally geodesic and intrinsically flat. Can it be isolated in the class of such tori? In other words, is it true that the torus belongs to a family of tori which are also totally geodesic and intrinsically flat?

5. Consider two co-compact lattices in the same Lie group. Is it true that they are bi-Lipschitz equivalent? This is obvious for Abelian groups, however is also true for non-amenable groups, and for lattices with equal co-volume.

If M^n is the universal cover of a Riemannian torus, does it admit an equivariant embedding in $\mathbb{R}^N$, where N is the optimal ("Nash") dimension for n? Can the universal cover of T^2 with some Riemannian metric be EMBEDDED into a compact domain in $\mathbb{R}^4$?

Consider a smooth surface in $\mathbb{R}^3$ with integral —curvature— being small. Is it true that nearby there is a C^2-smooth surface which is also intrinsically close?

There is a large basket of problems asking if specific functions (positive and separated from zero and infinity) are Jacobians of Lipschitz homeoomorphisms. With B. Klener and C. McMullen, we showed that such functions exist. (This suggests a very vague idea of non-smooth cohomologies of $\mathbb{R}^n$...). The examples are artificially constructed and very special, we do not know any characterization or any obstruction for a function to be a Jacobian of a Lipschitz local homeo. Even for locally constant functions taking just two values, say 1 and 2. For instance, a function which take value 1 inside a closed snowflake curve and 2 outside ("inflating" in a Lipschitz way a region surrounded by a snowflake curve).

For Finsler tori without conjugate points, a foliation of the unit tangent bundle into invariant tori (Heber's foliation) still exists, but its smoothness is unknown. Even for a single torus of the foliation.

10. A group is said to be *bounded* if it does not admit a bi-invariant (semi)-metric of infinite diameter. The questions below grew from a work by BIP. The *(semi)-norm* corresponding to d is given by $||g|| = d(g, id)$. The other way around, a bi-invariant norm can be defined in a usual way, and then it gives raise to a bi-invariant metric $d(g, h) := ||gh^{-1}||$. Thus a group is unbounded if it admits an unbounded semi-norm. This is a purely algebraic property, and now one can take her/his favorite group and check if it is bounded. Apart from obvious cases, this turns out to be surprisingly hard.

This notion is directly related to that of *bounded c-generation*: a group is *boundedly c-generated* if there exists a finite symmetric subset $G \subset \Gamma$ such that, for every $g \in \Gamma$, there is a number k such that g can be represented as a product $g = \tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_k$, where each $\tilde{g}_i$ is conjugate to some element of G. The smallest value of k is a norm (in many cases, universal, meaning that it is bounded iff the group is). Many interesting groups are c-generated by just one element and its inverse, say simple groups; and many groups of diffeomorphisms are simple. Whatever naive it seems, in any non-trivial case it is not so easy to check if a group is bounded. All abelian groups are obviously unbounded. $SL(2, \mathbb{Z})$ is unbounded but $SL(n, \mathbb{Z})$, $n > 2$ is bounded (this fact is based on a highly non-trivial theorem saying that these groups are boundedly generated).

Hyperbolic groups are unbounded. The group of diffeomorphisms (and homeomorphisms) isotopic to id of every compact odd-dimensional manifold is bounded, as well as that of the annulus (since it has a boundary, we consider diffeomorphisms identical near the boundary, otherwise there are obvious unbounded and not interesting norms). Even for the Mobius strip and 2-torus, the answer is unknown!

The absolute value of a quasi-morphism is a semi-norm. Now we are interested in finitely-presented groups, otherwise there are are examples like an infinite sum of $\mathbb{Z}/(2\mathbb{Z})$. This group obviously admits no unbounded quasi-morphisms but still carries a very nice unbounded norm: the number of 1s. (In some vague sense, for groups of diffeomorphism the property of being finitely presented corresponds to having a compact support). We however do not know even a single finitely presented group which has an unbounded semi-norm but no unbounded quasi-morphisms. Furthermore, every unbounded quasi-morphisms grows linearly along some one-parameter subgroup. Question: Does there exist a finitely-presented group which admits an unbounded semi-norm but every semi-norm grows sub-linearly (say, like a square root) along every one-parameter subgroup?

Consider a standard torus $\mathbb{T}^n$ and a C^k-diffeomorphism $\phi : \mathbb{T}^n \to \mathbb{T}^n$ isotopic to the identity. Can one give an estimate on the norm in C^k, C^{k-1} or such of the optimal vector field $V(x,t)$ such that the flow $\Phi_V(T)$ connects ϕ with the identity, that is $\Phi_V(0) = id$ and $\Phi_V(1) = \phi$? J. Lu and T. Ozuch have shown that in dim=2 such an estimate exists, but they could not obtain an explicit estimate. In their argument, they regard the torus as a family of circles and apply a certain curve shortening procedure to the images of these circles under ϕ, turning them again into closed geodesics (the problem and this idea came up in my discussions with L. Polterovich). There are no singularities that develop in the process, but how large curvature can appear in the process? There is a compactness argument. It is very likely that an explicit argument can be obtained, say by using the largest circles sitting inside the curve and touching it at various points. BTW, this argument by J. Lu and T. Ozuch does not fly for S^2 or surfaces of a higher genus.

It would be however a way more interesting if one can show that in dimensions say at least five such estimates cannot be obtained. Probably results of A. Nabutovski and S. Weinberg on the complexity of the Morse landscape of the length functional, and arguably more advanced results such as A. Hitchers' works on unbounded generation and F. Manin's works on introducing a new h-principle (about which I learned from Sh. Weinberger) or similar arguments can be used here.

We say that a metric space satisfies the FDP if for any two metrics d_1, d_2 such that $d_1(x,y)/d_2(x,y) \to 1$ as $d_1(x,y) \to \infty$, there is a $c \in \mathbb{R}$ such that for every x, y we have $|d_1(x,y) - d_2(x,y)| \leq c$. A result of mine (with an idea borrowed from F. Nazarov) is that the FDP holds for the lifts of length metrics on tori to their universal covers, which are topologically $\mathbb{R}^n$ with an isometric action of $\mathbb{Z}^n$. We call such metrics *periodic*. For every periodic metric d on $\mathbb{R}^n$ there exists a unique norm $||\cdot||$ such that $d(x,y)/||x-y|| \to 1$ as $d(x,y) \to \infty$. We call it *the stable norm*. The FDP property tells us that every periodic metric with a $\mathbb{Z}^n$ action lies within a bounded GH-distance from a normed space, and we have a rather misterious map from Reimannian metric on tori to convex bodies (the unit balls of the stable norms). This map is highly not surjective (to be discussed later) and probably not injective, though such cases should be rather special.

S. Krat showed the FDP for the discrete Heisenberg group with left-invariant metrics. One can show that all hyperbolic groups satisfy the FDP, this is easier than in the abelian case, but the proof is quite different. Now there are semi-hyperbolic groups in-between abelian and hyperbolic, and for them the question is widely open!

1There are plenty of problems asking what kind of norms (equivalently, their unit balls) can arise from Riemannian metrics on tori. I am mentioning just a few, which grew from the work by S. Ivanov, B. Kleiner, and myself (BIK). They mostly have to do with the smoothness of stable norms. A major problem is a generalization of a 2-dim result of V. Bangert: in higher dimensions, if the stable norm is smooth and strictly convex on an open region, is this true that the universal cover of the torus is foliated by minimizing geodesics?

In 2-dim case, the stable norm is smooth at irrational directions. Is this true in dimension 3? BKI constructed high-dimensional example whose stable norm is not smooth at almost all entirely irrational directions: the tangent cones to the unit sphere has 1-dim edges. These examples are of finite smoothness, though the smoothness grows with dimension. It is not clear if there are C^∞ examples and how large dimension is needed.

The unit ball of the stable norm of a periodic metric can be obtained as the limit of appropriate re-scalings of large balls in the metric. One can consider randomized versions of many questions here. For instance, consider the standard unit grid in the plane. This is a graph with all edges of length one and it carries a standard graph metric. Its stable norm is a diamond. Thus it has strong singularties at the vertices of the diamond, say at $(0, 1)$. The angle of the tangent cone is $\pi/2$ rather than π (as it is at smooth points). Let us do something similar to percolation, with a little change in order not to worry about connectivity. Now, with some probability p, let us change the length of each edge from 1 to, say, 0.1. We get a new metric and a new stable norm. The first question is: do we get the same norm with probability 1? Probably so. But anyways, the other question is: do we still see a singularity on the y-axis and, if so, how would the angle of the tangent cone change? With S .Ivanov, we did some computer simulations, but the results were inconclusive.

We have a smooth compact Riemannian manifold which is Anosov. The universal cover is, sure, topologically a Euclidean space. For any point x in the universal cover we can define the visibility measure μ_x on the ideal boundary by pushing forward the standard measure on the unit sphere in T_xM via geodesics. Our question is: are the measures μ_x mutually absolutely continuous for different x? If yes, are the Radon-Nikodym derivatives uniformly bounded away from 0 for any pair of points on a compact set? Note that this is not necessarily a Hadamard manifold: it may have regions of positive curvature. If yes, then, using existing results, this would help to resolve a long-standing problem (Michel's Conjecture).

Consider a surface Σ_1 whose boundary lies in a linear subspace and surrounds a domain Σ. Then is it true that $Area\Sigma_1 \geq Area\Sigma$, where $Area$ is the Busemann-Hausdorff surface area? With S. Ivanov, we proved this if $dim\Sigma = 2$ by explicitly presenting a calibrating form (a closed form ω such that $||\omega|| \leq 1$ and ω coincides with the ares form on Σ). Let $K \subset \mathbb{R}^2$ be a symmetric convex polygon, $f_1, \dots, f_n \colon \mathbb{R}^2 \to \mathbb{R}$ are linear functions such that $f_i|_K \leq 1$ for all i, and $p_1, \dots, p_n$ are nonnegative real numbers such that $\sum p_i = 1$. Then

$$\left| \sum_{1 \leq i < j \leq n} p_i p_j \, f_i \wedge f_j \right| \leq \sum_{1 \leq i < j \leq n} p_i p_j \, |f_i \wedge f_j| \leq \frac{1}{A(K)}.$$

In addition, if K is a convex $2n$-gon $a_1 a_2 \dots a_{2n}$, f_i are supporting functions of K corresponding to its sides (that is, such that $f_i = 1$ on $[a_i a_{i+1}]$), and $p_i = 2A(\triangle 0 a_i a_{i+1})/A(K)$, then the above inequalities turn into equalities. This can be viewed as probably a "new formula for the area of a centrally-symmetric convex polygon". This easily provides us with a calibrating form (after polyhedral approximations). However, we do not know how to construct such a form even for 3-dim surfaces. We suspect that a formula that works for regular polyhedra and the sphere would do the job, but we do not have one.