Talenti type results for linear and nonlinear Robin problems

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The classical isoperimetric inequality

In $\mathbb{R}^2$ the question was faced by Hurwitz, Minkowski, Lebesgue, Blaschke, Bonnesen, Lax...
In $\mathbb{R}^N$ by Tonelli, Schmidt, Radò, Cabrè...

Theorem (De Giorgi 1954) Let E be a Lebesgue measurable subset of $\mathbb{R}^N$, with finite measure, then

$$N\omega_N \left|E\right|^{1-\frac{1}{N}} \leq Per\left(E\right)$$

where equality holds iff E is a ball.
Here and throughout

- $|E|$ is the Lebesgue measure of E,
- ω_N is the measure of the unit ball in $\mathbb{R}^N$,
- $Per\left(E\right) = \sup\left\{\int_E \mathrm{div}\varphi dx,\ \varphi \in C_0^1\left(\mathbb{R}^N, \mathbb{R}^N\right),\ \|\varphi\|_\infty \leq 1\right\}$

Schwarz symmetrization

Let $\Omega \subset \mathbb{R}^N$, $u : x \in \Omega \to \mathbb{R}$ and $\Omega^\bigstar = B(0,r) : |\Omega| = |\Omega^\bigstar|$

$$\mu(t) = |\{x \in \Omega : |u(x)| > t\}|, \ t \geq 0$$

(distribution function of u)

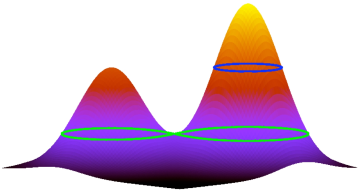
$$(\mu^{-1} \approx) \quad u^*(s) = \inf\{t \geq 0 : \mu(t) < s\}, \ s \in (0, |\Omega|]$$

(decreasing rearrangement of u)

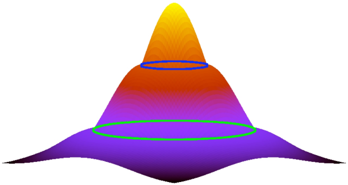
$$u^\bigstar(x) = u^*(\omega_N |x|^N), \ x \in \Omega^\bigstar$$

(Schwarz rearrangement of u)

$$\mu(t) = \int_t^{+\infty} \left(\int_{|u|=t} \frac{1}{|Du|} d\mathcal{H}^{N-1} \right) dt \Rightarrow -\mu'(t) = \int_{u=t} \frac{1}{|Du|} d\mathcal{H}^{N-1}$$



$u(x)$



$u^{\star}(x)$

Schwarz symmetrization

$u^\bigstar(x)$ is the unique function satisfying for each $t \geq 0$

$$\left\{x \in \Omega^\bigstar : u^\bigstar(x) > t\right\} = \{x \in \Omega : |u(x)| > t\}^\bigstar$$

Since u, u^* and $u^\bigstar$ are equidistributed we have

$$\|u\|_{L^p(\Omega)} = \left\|u^\bigstar\right\|_{L^p(\Omega^\bigstar)} = \|u^*\|_{L^p(0,|\Omega|)} \quad \forall p \in [1, +\infty] \, .$$

Moreover

$$\int_E |u| \, dx \leq \int_0^{|E|} u^* ds, \quad \forall E \subset \Omega$$

and

$$\int_\Omega |fg| \, dx \leq \int_0^{|\Omega|} f^* g^* ds$$

(Hardy-Littlewood inequality)

Schwarz rearrangement leaves the L^p-norms of u unaltered but what happens to the L^p-norms of $|\nabla u|$?

Pólya-Szegő principle Let $u \in W_0^{1,p}(\Omega)$ be a nonnegative function, then

$$(1) \qquad \int_\Omega |\nabla u|^p \, dx \geq \int_{\Omega^\star} \left|\nabla u^\star\right|^p dx, \text{ for any } p \in [1, \infty) .$$

Furthermore [J.E. Brothers - W.P. Ziemer, 1988] if

$$\left|\{Du = 0\} \cap \left\{u^{-1}\left(0, \sup u\right)\right\}\right| = 0,$$

then equality holds in (1) if and only if $u = u^\star$, modulo translation.

Remark The Pólya-Szegő principle immediately implies the Faber-Krahn inequality. That is

$$\lambda_1^{\mathrm{D}}(\Omega) \geq \lambda_1^{\mathrm{D}}(\Omega^\star),$$

where $\lambda_1^{\mathrm{D}}(\Omega)$ denotes the first eigenvalue of the Dirichlet-Laplacian in Ω.

Lorentz spaces

Let $0 < p < \infty$ and $0 < q \leq \infty$, the Lorentz space $L^{p,q}(\Omega)$ consists of all functions u such that the following quantity

$$\|u\|_{L^{p,q}(\Omega)} = \begin{cases} p^{\frac{1}{q}} \left(\int_0^\infty t^q \mu(t)^{\frac{q}{p}} \frac{dt}{t} \right)^{\frac{1}{q}} & 0 < q < \infty \\ \sup\limits_{t>0} \left(t^p \mu(t) \right) & q = \infty \end{cases}$$

is finite.

Remark Whenever $p = q$ Lorentz spaces coincide with Lebesgue spaces L^p, since by Cavalieri's Principle we have

$$\|u\|_{L^{p,p}(\Omega)} = \|u\|_{L^p(\Omega)} = p^{\frac{1}{p}} \left(\int_0^\infty t^{p-1} \mu(t) dt \right)^{\frac{1}{p}}$$

Talenti's Theorem in its simplest form

Theorem Let $\Omega \subset \mathbb{R}^N$, $0 \le f(x) \in L^p(\Omega)$ with $p > 1$ if $N = 2$ and $p = \frac{2N}{N+2}$ if $N > 2$. Consider the problems

$$\begin{cases} -\Delta u = f(x) & \text{in} \quad \Omega \\ \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Delta v = f^\bigstar(x) & \text{in} \quad \Omega^\bigstar \\ \\ v = 0 & \text{on} \quad \partial\Omega^\bigstar. \end{cases}$$

Then

$$u^\bigstar(x) \le v(x) \text{ in } \Omega^\bigstar.$$

Remark In the Talenti's Theorem it is crucial that

$$\partial\{u > t\} \cap \partial\Omega = \emptyset \quad \forall t > 0.$$

A few references on symmetrization techniques and applications

Pioneering results

- Weinberger, 1962
- Maz'ja, 1969
- Talenti, 1976

Survey papers

- Trombetti, 2000
- Talenti, 2016

Monographs

- Hardy - Littlewood - Pólya, 1952
- Bandle, 1980
- Kawohl, 1985
- Kesavan, 2006
- Henrot, 2006
- Cianchi, 2010
- Henrot, 2017
- Baernstein II, 2019

Talenti's type results for the Robin problem

Consider

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a Lipschitz domain in $\mathbb{R}^N$, $N \geq 2$, $\beta > 0$, ν denotes the outer unit normal to $\partial\Omega$ and $0 \leq f(x) \in L^2(\Omega)$.

Remark Now it may happens that

$$\partial\{u > t\} \cap \partial\Omega \neq \emptyset \quad \text{for some } t > 0.$$

Talenti's type results for the Robin problem

Theorem [Alvino - Nitsch - Trombetti, to appear on CPAM]. Let v be the solution of

$$\begin{cases} -\Delta v = f^{\bigstar}(x) & \text{in } \Omega^{\bigstar} \\ \dfrac{\partial v}{\partial \nu} + \beta v = 0 & \text{on } \partial\Omega^{\bigstar}. \end{cases}$$

Then

$$\|u\|_{L^{p,1}(\Omega)} \leq \|v\|_{L^{p,1}(\Omega^{\bigstar})} \text{ for all } 0 < p \leq \frac{N}{2N-2}$$

and

$$\|u\|_{L^{2p,2}(\Omega)} \leq \|v\|_{L^{2p,2}(\Omega^{\bigstar})} \text{ for all } 0 < p \leq \frac{N}{3N-4}$$

Finally if $N = 2$ and $f(x) \equiv 1$, then

$$u^{\bigstar}(x) \leq v(x) \text{ in } \Omega^{\bigstar}.$$

Subsequent developments

- Alvino - C. - Nitsch - Trombetti, 2021. [Comparison results for the Robin Laplacian, with $\beta = \beta(x)$]
- Amato - Gentile - Masiello, 2022. [Comparison results for the Robin $p-$Laplacian, with $\beta = C$]
- Amato -Masiello - Nitsch - Trombetti, 2022 [Asymptotic as $p \to +\infty$, with $\beta = C$]
- C. - Gavitone - Nitsch - Trombetti, 2022. [Comparison results for the Hermite operator with Robin boundary conditions, with $\beta = C$]
- Alvino - C. - Nitsch - Trombetti, 2022 [Comparison results for the Robin Laplacian, with $\beta = \beta(x)$, via weighted rearrangement]
- Amato - C. - Gentile, 2022. [Comparison results for the Robin $p-$Laplacian, with $\beta = \beta(x)$, via weighted rearrangement]

Talenti's type result when $\beta = \beta(x)$: an approach via Schwarz symmetrization

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \beta(x) u = 0 & \text{on } \partial\Omega \end{cases} \qquad (P) \qquad \begin{cases} -\Delta v = f^\star(x) & \text{in } \Omega^\star \\ \dfrac{\partial v}{\partial \nu} + \widehat{\beta} v = 0 & \text{on } \partial\Omega^\star \end{cases} \qquad (P^\star)$$

where Ω is a Lipschitz domain in $\mathbb{R}^N$, $N \geq 2$, ν denotes the outer unit normal to $\partial\Omega$,

$$\beta(x) : \partial\Omega \to \mathbb{R} \text{ such that } 0 < m < \beta(x) < M < +\infty \text{ a.e. in } \Omega,$$

$$0 \leq f(x) \in L^2(\Omega),$$

$\widehat{\beta}$ is the positive constant defined by the following relation

$$\frac{Per(\Omega^\sharp)}{\widehat{\beta}} = \left(\int_{\partial\Omega^\sharp} \frac{1}{\widehat{\beta}} d\mathcal{H}^{N-1}(x) \right) = \int_{\partial\Omega} \frac{1}{\beta(x)} d\mathcal{H}^{N-1}(x).$$

Talenti's type result when $\beta = \beta(x)$: an approach via Schwarz symmetrization

Theorem [Alvino - C. - Nitsch - Trombetti, 2021, JMPA] Let u and $v = v^\star$ be the solutions to Problem (P) and $(P^\star)$, respectively. Then, when $N = 2$, we have

$$\|u\|_{L^1(\Omega)} \leq \|v\|_{L^1(\Omega^\star)}.$$

While for $N \geq 3$

$$\|u\|_{L^1(\Omega)} \leq \|v\|_{L^1(\Omega^\star)},$$

provided

$$\int_E f(x)dx \leq \frac{|E|^{1-\frac{2}{N}}}{|\Omega|^{1-\frac{2}{N}}} \int_\Omega f(x)dx$$

for all measurable $E \subseteq \Omega$.
Finally for $N = 2$ and $f \equiv 1$ we have

$$u^\star(x) \leq v(x) \quad x \in \Omega^\star.$$

Talenti's type result when $\beta = \beta(x)$: an approach via weighted symmetrization

$$(P) \qquad \begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right) = f(x)\,|x|^{\delta} & \text{in } \Omega \\ \\ |\nabla u|^{p-2}\,\dfrac{\partial u}{\partial \nu} + \beta(x)u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p > 1$, $N \geq 2$, ν denotes the outer unit normal to $\partial\Omega$ and Ω is a bounded and Lipschitz domain in $\mathbb{R}^N$, with $0 \notin \partial\Omega$. We will assume that

$$(H_1) \qquad p \geq N,$$

$$(H_2) \qquad -N < \delta < 0,$$

$$(H_3) \qquad m = \inf_{\partial\Omega} \beta(x) > 0 \quad \text{and} \quad M = \sup_{\partial\Omega} \beta(x) < +\infty,$$

$$(H_4) \qquad 0 \leq f(x) \in L^{p'}(\Omega, |x|^{\delta}\,dx), \quad \text{where} \quad p' = \frac{p}{p-1}.$$

The symmetrized problem

$$(P^\sharp) \qquad \begin{cases} -\text{div}\left(|\nabla v|^{p-2}\nabla v\right) = f^\sharp(x)\,|x|^\delta & \text{in } \Omega^\sharp \\[2ex] |\nabla v|^{p-2}\dfrac{\partial v}{\partial \nu} + \widetilde{\beta}\left(r^\sharp\right)^{\frac{\delta}{p'}} v = 0 & \text{on } \partial\Omega^\sharp, \end{cases}$$

where

$$\widetilde{\beta} = \inf_{\partial\Omega} \beta(x)\,|x|^{-\frac{\delta}{p'}} \qquad \left(\Rightarrow \beta(x) \geq \widetilde{\beta}\cdot|x|^{\frac{\delta}{p'}}\right)$$

$\Omega^\sharp$ is the ball centered at the origin of radius $r^\sharp$, with $r^\sharp$:

$$\left|\Omega^\sharp\right|_\delta = \int_{\Omega^\sharp} |x|^\delta\,dx = \int_\Omega |x|^\delta\,dx = |\Omega|_\delta\,,$$

$f^\sharp(x)$ is the unique radial and radially decreasing function such that

$$|\{x\in\Omega : f(x) > t\}|_\delta = \left|\left\{x\in\Omega : f^\sharp(x) > t\right\}\right|_\delta \quad \text{for any } t\geq 0.$$

Main results

Theorem 1 [Alvino - C. - Nitsch - Trombetti] and [Amato - C. - Gentile]. Let u and v be the solutions to problems (P) and $(P^\sharp)$, respectively. Then

$$\int_\Omega |u(x)|\,|x|^\delta\,dx \le \int_{\Omega^\sharp} |v(x)|\,|x|^\delta\,dx$$

and for any $p \ge N$

$$\int_\Omega |u(x)|^p\,|x|^\delta\,dx \le \int_{\Omega^\sharp} |v(x)|^p\,|x|^\delta\,dx.$$

Theorem 2 [Alvino - C. - Nitsch - Trombetti] and [Amato - C. - Gentile]. Suppose that $f(x) \equiv 1$ in Ω. If either $p = N = 2$ or $p > 2$, $N \ge 2$ and

$$\delta \le -N + \frac{p-N}{p-2}$$

then

$$u^\sharp(x) \le v(x) \;\text{ a.e. in } \Omega^\sharp.$$

Sketch of the proof of Theorem 2 in the simplest case

If

$$N = p = 2, \quad \beta(x) = \beta \cdot |x|^{\delta/2}, \quad f(x) \equiv 1, \quad \delta \in (-2, 0)$$

$$(P) \qquad \begin{cases} -\Delta u = |x|^{\delta} & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \beta\, |x|^{\delta/2}\, u = 0 & \text{on } \partial\Omega \end{cases}$$

$$(P^\sharp) \qquad \begin{cases} -\Delta v = |x|^{\delta} & \text{in } \Omega^\sharp \\ \dfrac{\partial v}{\partial \nu} + \beta\, \left(r^\sharp\right)^{\delta/2} v = 0 & \text{on } \partial\Omega^\sharp. \end{cases}$$

Then

$$u^\sharp(x) \le v(x) \text{ in } \Omega^\sharp.$$

A few notation

Let $t \geq 0$.

$$U_t = \{x \in \Omega\colon u(x) > t\}, \;\; \partial U_t^{\text{int}} = \partial U_t \cap \Omega, \;\; \partial U_t^{\text{ext}} = \partial U_t \cap \partial\Omega$$

$$\mu(t) = |U_t|_\delta := \int_{U_t} |x|^\delta \, dx \quad \text{and} \quad P_u(t) = P_{\frac{\delta}{2}}(U_t) := \int_{\partial U_t} |x|^{\delta/2} \, d\mathcal{H}^1$$

$$u^*(s) = \inf\{t \geq 0 : \mu(t) < s\}, \;\; s \in (0, |\Omega|_\delta]$$

$$u^\sharp(x) = u^*\left(|B(0,|x|)|_\delta\right) = u^*\left(\frac{2\pi}{2+\delta}|x|^{2+\delta}\right)$$

$u^\sharp(x)$ is the unique radial and radially decreasing function such that

$$|\{x \in \Omega : |u(x)| > t\}|_\delta = \left|\left\{x \in \Omega : u^\sharp(x) > t\right\}\right|_\delta \;\; \text{for any } t \geq 0.$$

Finally

$$V_t = \{x \in \Omega^\sharp\colon v(x) > t\}, \;\; \phi(t) = |V_t|_\delta \;\; \text{and} \quad P_v(t) = P_{\frac{\delta}{2}}(V_t)$$

A weighted isoperimetric inequality

Theorem [Chiba - Horiuchi (2015)] and [Alvino - Brock - C. - Mercaldo - Posteraro (2017)]. Let G be a Lebesgue measurable subset in $\mathbb{R}^2$ and let $\delta \in (-2, 0)$. Define

$$|G|_\delta = \int_G |x|^\delta \, dx \quad \text{and} \quad P_{\frac{\delta}{2}}(G) = \int_{\partial\Omega} |x|^{\frac{\delta}{2}} \, d\mathcal{H}^1.$$

Then

$$P_{\frac{\delta}{2}}(G) \geq P_{\frac{\delta}{2}}(G^\sharp),$$

where $G^\sharp = B(0, r^\sharp)$ with $r^\sharp > 0$:

$$|G|_\delta = \left| G^\sharp \right|_\delta .$$

Remark Note that the isoperimetric inequality above can be written equivalently as follows

$$P^2_{\frac{\delta}{2}}(G) \geq 2\pi \, (\delta + 2) \, |G|_\delta \, .$$

$0 \leq u_m \leq v_m$

Lemma 1 The following inequalities hold true

$$0 \leq u_m \leq v_m,$$

where

$$u_m := \min_{\overline{\Omega}} u, \quad v_m := \min_{\overline{\Omega^{\#}}} v.$$

Proof of $u_m \leq v_m$.

$$v_m P_{\frac{\delta}{2}}(\Omega^\sharp) = \int_{\partial\Omega^\sharp} v(x)\,|x|^{\delta/2}\, d\mathcal{H}^1 = -\frac{1}{\beta}\int_{\partial\Omega^\sharp} \frac{\partial v}{\partial \nu} d\mathcal{H}^1 = -\frac{1}{\beta}\int_{\partial\Omega^\sharp} \Delta v dx$$

$$= \frac{1}{\beta}\int_{\Omega^\sharp} |x|^{\delta}\, dx = \frac{1}{\beta}\int_{\Omega} |x|^{\delta}\, dx = -\frac{1}{\beta}\int_{\partial\Omega} \Delta u dx = -\frac{1}{\beta}\int_{\partial\Omega} \frac{\partial u}{\partial \nu} d\mathcal{H}^1$$

$$= \int_{\partial\Omega} u\,|x|^{\delta/2}\, d\mathcal{H}^1 \geq u_m P_{\frac{\delta}{2}}(\Omega) \geq u_m P_{\frac{\delta}{2}}(\Omega^\sharp).$$

Proof of $0 \leq u_m$. Use $u^- = \max\{0, -u\}$ as test function in (P).

Some auxiliary results

Lemma 2.1 For all $t \geq v_m$ we have

$$\int_0^t \tau \left(\int_{\partial U_t^{\text{ext}}} \frac{|x|^{\delta/2}}{u(x)} d\mathcal{H}^1 \right) d\tau \leq \frac{|\Omega|_\delta}{2\beta}.$$

Proof Clearly

$$\int_0^t \tau \left(\int_{\partial U_t^{\text{ext}}} \frac{|x|^{\delta/2}}{u(x)} d\mathcal{H}^1 \right) d\tau \leq \int_0^\infty \tau \left(\int_{\partial U_t^{\text{ext}}} \frac{|x|^{\delta/2}}{u(x)} d\mathcal{H}^1 \right) d\tau$$

$\Downarrow$ (Fubini's Theorem) $\Downarrow$

$$= \int_{\partial\Omega} \left(\int_0^{u(x)} \tau d\tau \right) \frac{|x|^{\delta/2}}{u(x)} d\mathcal{H}^1 = \frac{1}{2} \int_{\partial\Omega} u(x) \, |x|^{\delta/2} \, d\mathcal{H}^1$$

Some auxiliary results

$$\Downarrow \left(u\,|x|^{\delta/2} \;=-\frac{1}{\beta}\frac{\partial u}{\partial \nu} \;\text{ on }\; \partial\Omega \right) \Downarrow$$

$$\frac{1}{2}\int_{\partial\Omega} u(x)\,|x|^{\delta/2}\,d\mathcal{H}^1 = -\frac{1}{2\beta}\int_{\partial\Omega}\frac{\partial u}{\partial \nu}d\mathcal{H}^1$$

$\Downarrow$ (Divergence Theorem) $\Downarrow$

$$-\frac{1}{2\beta}\int_{\partial\Omega}\frac{\partial u}{\partial \nu}d\mathcal{H}^1 = -\frac{1}{2\beta}\int_{\Omega}\Delta u dx$$

$$\Downarrow \left(-\Delta u = |x|^{\delta} \;\text{ in }\; \Omega \right) \Downarrow$$

$$-\frac{1}{2\beta}\int_{\Omega}\Delta u dx = \frac{1}{2\beta}\int_{\Omega}|x|^{\delta}\,dx = \frac{|\Omega|_{\delta}}{2\beta}.$$

Some auxiliary results

Lemma 2.2 For all $t \geq v_m$ we have

$$\int_0^t \tau \left(\int_{\partial V_t \cap \partial \Omega^\sharp} \frac{|x|^{\delta/2}}{v(x)} d\mathcal{H}^1 \right) d\tau = \frac{|\Omega|_\delta}{2\beta}.$$

Proof

$$v \equiv v^\sharp \Rightarrow \begin{cases} \partial V_t = \partial \Omega^\sharp & \forall\ t \in [0, v_m] \\ \partial V_t \cap \partial \Omega^\sharp = \emptyset & \forall\ t \in (v_m, +\infty) \end{cases} \Rightarrow$$

$$\int_0^t \tau \left(\int_{\partial V_t \cap \partial \Omega^\sharp} \frac{|x|^{\delta/2}}{v(x)} d\mathcal{H}^1 \right) d\tau = \int_0^{+\infty} \tau \left(\int_{\partial V_t \cap \partial \Omega^\sharp} \frac{|x|^{\delta/2}}{v(x)} d\mathcal{H}^1 \right) d\tau = \frac{|\Omega|_\delta}{2\beta}.$$

Some auxiliary results

Lemma 3.1 It holds that

$$\pi\,(\delta+2)\,\tau^2 \le \int_0^\tau \left(-t\mu'(t)\right) dt + \frac{|\Omega|_\delta}{2\beta}, \quad \forall \tau \ge v_m$$

Proof

$$-\int_{\partial U_t^{\text{ext}}} \frac{\partial u}{\partial \nu} d\mathcal{H}^1 + \int_{\partial U_t^{\text{int}}} |\nabla u|\, d\mathcal{H}^1 = -\int_{U_t} \Delta u dx = \int_{U_t} |x|^\delta\, dx = \mu(t)$$

$$\Downarrow \left(-\frac{\partial u}{\partial \nu} = \beta u\, |x|^{\delta/2} \quad \text{on} \quad \partial\Omega_t^{\text{ext}} \right) \Downarrow$$

$$\int_{\partial U_t^{\text{ext}}} \beta u\, |x|^{\delta/2}\, d\mathcal{H}^1 + \int_{\partial U_t^{\text{int}}} |\nabla u|\, d\mathcal{H}^1 = \mu(t)$$

Summarizing, setting

$$g(x) := \begin{cases} |\nabla u| & \text{if} \quad x \in \partial U_t^{\text{int}} \\ \beta u\, |x|^{\delta/2} & \text{if} \quad x \in \partial {U_t}^{\text{ext}}, \end{cases}$$

Some auxiliary results

we have shown that

$$\int_{\partial U_t} g(x)\, d\mathcal{H}^1 = \mu(t).$$

$$2\pi\,(\delta+2)\,\mu(t) \le P_u^2(t) = \left(\int_{\partial U_t} \sqrt{g(x)}\,\frac{|x|^{\delta/2}}{\sqrt{g(x)}}\, d\mathcal{H}^1\right)^2$$

$$\le \left(\int_{\partial U_t} g(x)\, d\mathcal{H}^1\right)\left(\int_{\partial U_t} \frac{|x|^{\delta}}{g(x)}\, d\mathcal{H}^1\right)$$

$$= \mu(t)\left(\int_{\partial U_t^{\text{int}}} \frac{|x|^{\delta}}{|\nabla u|}\, d\mathcal{H}^1 + \frac{1}{\beta}\int_{\partial U_t^{\text{ext}}} \frac{|x|^{\delta/2}}{u(x)}\, d\mathcal{H}^1\right)$$

$$= \mu(t)\left(-\mu'(t) + \frac{1}{\beta}\int_{\partial U_t^{\text{ext}}} \frac{|x|^{\delta/2}}{u(x)}\, d\mathcal{H}^1\right)$$

Some auxiliary results

Hence, multiplying both sides of the inequality by t, we get

$$2\pi\,(\delta+2)\,t \le \left(-\mu'(t)\right)\,t + \frac{t}{\beta}\int_{\partial U_t^{\text{ext}}} \frac{|x|^{\delta/2}}{u(x)}\,d\mathcal{H}^1.$$

Integrating on $(0,\tau)$, with $\tau \ge v_m$, and using Lemma 2.1 we get

$$\pi\,(\delta+2)\,\tau^2 \le \int_0^\tau \left(-t\mu'(t)\right)\,dt + \frac{|\Omega|_\delta}{2\beta},$$

that is Lemma 3.1.

Lemma 3.2 It holds that

$$\pi\,(\delta+2)\,\tau^2 = \int_0^\tau \left(-t\phi'(t)\right)\,dt + \frac{|\Omega|_\delta}{2\beta}, \quad \forall \tau \ge v_m.$$

Proof It is again a consequence of the fact that $v \equiv v^\sharp$.

The pointwise estimate: $u^\sharp(x) \leq v(x)$ in $\Omega^\sharp$

Proof of Theorem 2 Lemmata 3.1 and 3.2 imply

$$\int_0^\tau \left(-t\phi'(t)\right) dt \leq \int_0^\tau \left(-t\mu'(t)\right) dt, \quad \tau \geq v_m$$

$\Downarrow$ (Integrating by parts) $\Downarrow$

$$\mu(\tau) \leq \phi(\tau), \quad \tau \geq v_m$$

$$\Downarrow \; (0 \leq u_m \leq v_m) \; \Downarrow$$

$$\mu(\tau) \leq \phi(\tau) \equiv |\Omega|_\delta \,, \quad 0 \leq \tau < v_m$$

$$\Rightarrow \mu(\tau) \leq \phi(\tau), \; \tau \geq 0$$

$$\Rightarrow u^\sharp(x) \leq v(x) \text{ in } \Omega^\sharp.$$

A Faber-Krahn inequality

Assume that $N = p = 2$, $\beta(x) = \beta \cdot |x|^{\delta/2}, \delta \in (-2, 0)$. Let $\lambda_{1,\delta}(\Omega)$ and $\lambda_{1,\delta}(\Omega^\sharp)$ be the first eigenvalues of the problems

$$\begin{cases} -\Delta u = \lambda(\Omega)\, |x|^{\delta}\, u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \beta\, |x|^{\delta/2}\, u = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Delta v = \lambda(\Omega^\sharp)\, |x|^{\delta}\, v & \text{in } \Omega^\sharp \\ \dfrac{\partial u}{\partial \nu} + \beta\, \left(r^\sharp\right)^{\delta/2} v = 0 & \text{on } \partial\Omega^\sharp. \end{cases}$$

We have

$$\lambda_{1,\delta}(\Omega) = \min_{w \in H^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla w|^2\, dx + \beta \int_{\partial\Omega} w^2\, |x|^{\delta/2}\, d\mathcal{H}^1}{\displaystyle\int_\Omega w^2\, |x|^{\delta}\, dx}.$$

A Faber-Krahn inequality

Theorem It holds that

$$\lambda_{1,\delta}(\Omega) \geq \lambda_{1,\delta}(\Omega^\sharp).$$

Proof Let u_1 be an eigenfunction corresponding to $\lambda_{1,\delta}(\Omega)$:

$$\begin{cases} -\Delta u_1 = \lambda_{1,\delta}(\Omega)\,|x|^\delta\, u_1 & \text{in } \Omega \\ \dfrac{\partial u_1}{\partial \nu} + \beta\,|x|^{\delta/2}\, u_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Denote by z the solution to

$$\begin{cases} -\Delta z = \lambda_{1,\delta}(\Omega)\,|x|^\delta\, u_1^\sharp & \text{in } \Omega^\sharp \\ \dfrac{\partial z}{\partial \nu} + \beta\,\left(r^\sharp\right)^{\delta/2} z = 0 & \text{on } \partial\Omega^\sharp. \end{cases}$$

A Faber-Krahn inequality

Cauchy-Schwarz inequality $+$ $L^2(\Omega, |x|^\delta\, dx)$ comparison

$$\Downarrow$$

$$\int_{\Omega^\sharp} u_1^\sharp z\, |x|^\delta\, dx \le \left(\int_{\Omega^\sharp} \left(u_1^\sharp\right)^2 |x|^\delta\, dx \right)^{\frac{1}{2}} \left(\int_{\Omega^\sharp} z^2\, |x|^\delta\, dx \right)^{\frac{1}{2}} \le \int_{\Omega^\sharp} z^2\, |x|^\delta\, dx$$

$$\Downarrow$$

$$\lambda_{1,\delta}(\Omega) = \frac{\displaystyle\int_{\Omega^\sharp} |\nabla z|^2\, dx + \beta \int_{\partial\Omega^\sharp} z^2\, |x|^{\delta/2}\, d\mathcal{H}^1}{\displaystyle\int_{\Omega^\sharp} u_1^\sharp z\ \ |x|^\delta\, dx}$$

$$\ge \frac{\displaystyle\int_{\Omega^\sharp} |\nabla z|^2\, dx + \beta \int_{\partial\Omega^\sharp} z^2\, |x|^{\delta/2}\, d\mathcal{H}^1}{\displaystyle\int_{\Omega^\sharp} z^2\, |x|^\delta\, dx} \ge \lambda_{1,\delta}(\Omega^\sharp).$$

THANK YOU !!