Mean inequalities for symmetrizations of convex bodies

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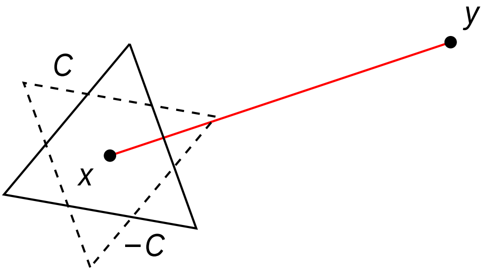
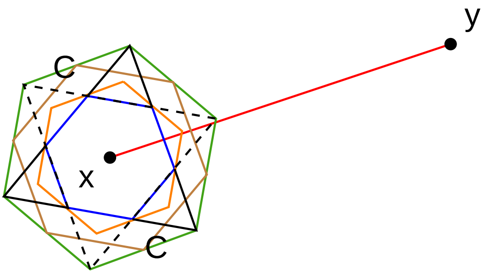
Motivation: Average Length


y
C
x
$-C$



$$\max\{\|x - y\|_C, \|y - x\|_C\} = \|x - y\|_{C \cap (-C)}$$

$$1/2\,(\|x - y\|_C + \|y - x\|_C) = \|x - y\|_{\left(\frac{C - C}{2}\right)}$$

$$R(\{x, y\}; C) = \|x - y\|_{\frac{C - C}{2}}$$

$$\geq \|x - y\|_{\mathrm{conv}(C \cup (-C))}$$

AM-HM mean inequality

Let $a, b > 0$. Then

$$\min\{a, b\} \quad \frac{2ab}{a+b} \quad \frac{a+b}{2} \quad \max\{a, b\}$$

AM-HM mean inequality

Let $a, b > 0$. Then

$$\min\{a, b\} \leq \frac{2ab}{a+b} \leq \frac{a+b}{2} \leq \max\{a, b\}$$

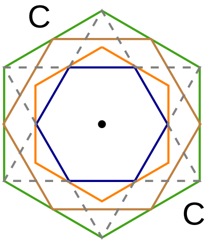
Let A, B be convex sets with $0 \in \mathrm{int}(A) \cap \mathrm{int}(B)$. Firey '61 has shown

$$A \cap B \subseteq \left(\frac{A^\circ + B^\circ}{2}\right)^\circ \subseteq \frac{A+B}{2} \subseteq \mathrm{conv}(A \cup B).$$

Always Optimal

Let C be a convex set with 20 int(C). Then

$$C \backslash \ (\quad C) \quad {}^{\text{opt}} \quad \frac{C \quad + (\quad C)}{2}^{!} \qquad \frac{C + (\quad C)}{2} \quad {}^{\text{opt}} \ \text{conv}(C\, [\ (\quad C)):$$



$$C \backslash \;(\;\; C) \quad {}^{\text{opt}} \quad \frac{C \;\; + (\;\;\; C)}{2}^{\,!} \qquad \frac{C + (\;\;\; C)}{2} \quad {}^{\text{opt}} \; \text{conv}(C \,[\;\; (\;\;\; C)) :$$

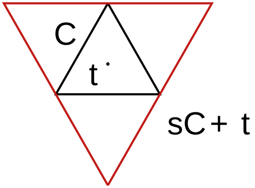
What about:

$$\frac{C \;\; + (\;\;\; C)}{2}^{\,!} \qquad {}_{?}\, \frac{C + (\;\;\; C)}{2}$$

$$C \backslash \;(\;\; C) \quad {}^{?} \, \text{conv}(C \,[\;\; (\;\;\; C))$$

Minkowski Asymmetry

$$s(C) := \inf\{\ \ > 0 \,|\, C \ \ (\ \ C) + t \text{ for some } t \in \mathbb{R}\} = R(C;\ \ C):$$



$c =$ Minkowski center: $C \ \ c \ \ s(C)(\ \ (C \ \ c)):$

$0 =$ Minkowski center: C is Minkowski centered.

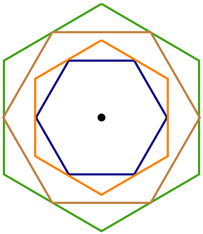
Theorem

Let S be a Minkowski centered regular n-simplex. Then

(i) S \ (S) $^{\text{opt}}$ conv(S [(S)), if n is odd,

(ii) S \ (S) $^{\text{opt}}$ $\frac{n}{n+1}$conv(S [(S)), if n is even, and

(iii) $\frac{S \quad S}{2}$ opt $\frac{n(n+2)}{(n+1)^2}$ $\frac{S \quad S}{2}$, if n is even.



Reverse mean inequalities

Theorem

Let C 2 K^n be Minkowski centered. Then

(i) conv(C [(C)) opt s(C)(C \ (C)),

(ii) conv(C [(C)) opt $\frac{2s(C)}{s(C)+1} \frac{C \quad C}{2}$,

(iii) $\frac{C \quad C}{2}$ opt $\frac{2s(C)}{s(C)+1}$(C \ (C)),

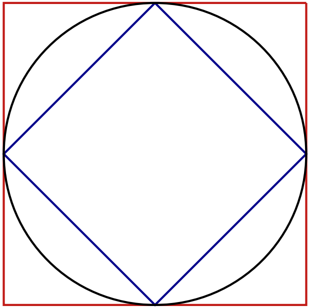
(iv) $\frac{C \quad C}{2}$ opt $\frac{s(C)+1}{2}$(C \ (C)),

(v) conv(C [(C)) opt $\frac{s(C)+1}{2} \quad \frac{C \quad C}{2}$.

(vi) $\frac{C \quad C}{2} \quad \frac{s(C)+1}{2} \quad \frac{C \quad C}{2}$, and for all s2 [1; n] 9 a Minkowski centered C2 K^n with s(C) = s, s.th. containment is optimal.

Let $x \in \mathbb{R}^n$. Then

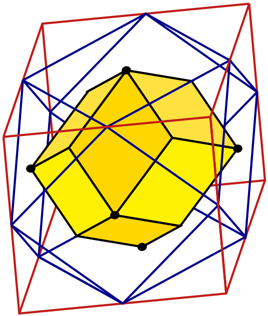
$$\|x\|_\infty \le \|x\|_2 \le \|x\|_1 \quad \text{and} \quad \|x\|_1 \le \sqrt{n}\|x\|_2 \le n\|x\|_\infty$$



$$kxk_{conv(C[\ (\ \ C))} \quad k\ \ xk_{\frac{C\ \ \ C}{2}} \quad k\ \ xk_{\left(\frac{C\ \ \ C}{2}\right)} \quad k\ \ xk_{C\backslash\ (\ \ C)}$$

$$kxk_{C\backslash\ (\ \ C)} \quad \frac{s+1}{2} kxk_{\frac{C\ \ \ C}{2}} \quad skxk_{conv(C[\ (\ \ C))}$$

$$\text{or } kxk_{C\backslash\ (\ \ C)} \quad \frac{2s}{s+1} kxk_{\left(\frac{C\ \ \ C}{2}\right)} \quad skxk_{conv(C[\ (\ \ C))}$$



Theorem

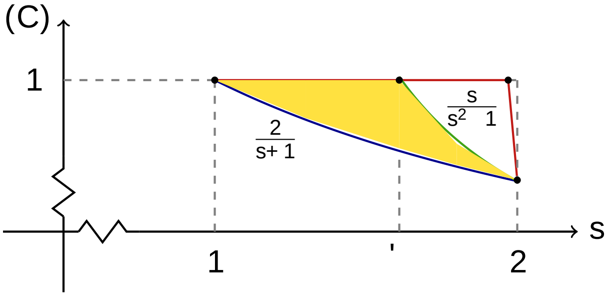
(i) C \ (C) ^{opt} conv(C [(C)) ()

(ii) $\frac{1}{2}$(C C)) ^{opt} $\frac{1}{2}$(C C) ()

(iii) 9p; p 2 bd(C), parallel halfspaces $H_{a;}$, $H_{\ a;}$ supporting C at p, p, respectively.

Let C 2 K ^{n}, (C) > 0 s.th. C \ (C) ^{opt} (C) conv(C [(C)).
For s 2 [1; n] we de ne

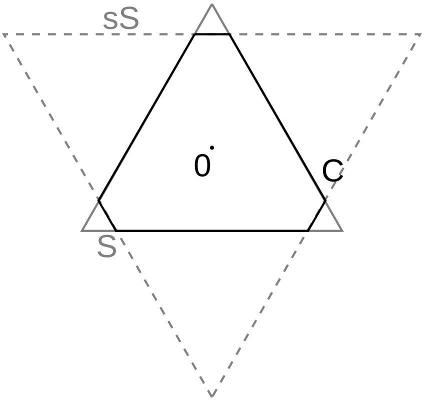
$$_{1n2}(s) := \inf \; n \sup f \quad (C) : C \; 2 \; K^n \text{ Minkowski centered}; s(C) = sg:$$



For n = 2: $_1(s) = \frac{2}{s+1}$. $_2(s) = 1$ for s ' := $\frac{1+ p\,\overline{5}}{2}$;
$_2(s)$ $\frac{s}{s^2 \;\; 1}$ for s ' , $_2$ $\frac{26s^2 + 36s + 34}{18s^2 \;\; 24s \;\; 21}$ for s > ' .

For $C = S \cap (-sS)$ with S a regular Minkowski centered simplex hold

$$C \cap (-C) \subset^{opt} \frac{2}{s+1} \operatorname{conv}(C \cup (-C))$$



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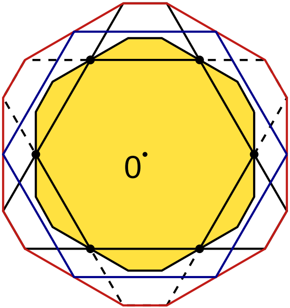
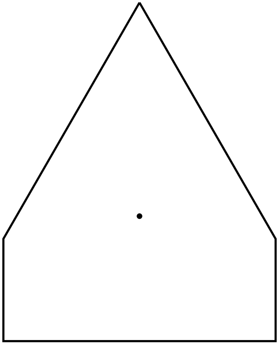


$$C \cap (-C) \subset^{opt} \max\{1, \frac{s}{s^2-1}\} \operatorname{conv}(C \cup (-C))$$



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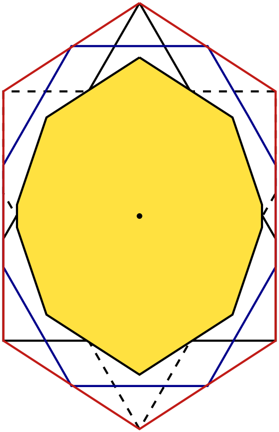
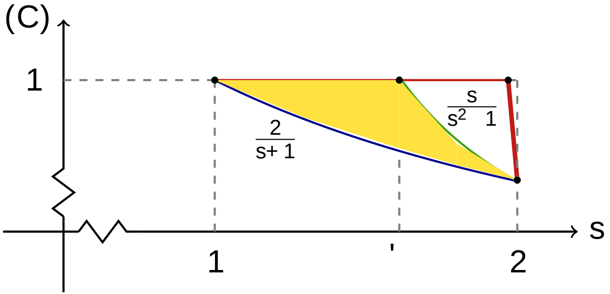


Let C 2 K ^{n}, (C) > 0 s.th. C \ (C) ^{opt} (C) conv(C [(C)).
For s 2 [1; n] we de ne

$$_{1n2}(s) := \inf \; n \sup f \;\; (C) : C\ 2\ K^n \text{ Minkowski centered}; s(C) = \; sg:$$



For n = 2: $_1(s) = \frac{2}{s+1}$. $_2(s) = 1$ for s ' := $\frac{1+\sqrt{5}}{2}$;
$_2(s)$ $\frac{s}{s^2 \;\; 1}$ for s ' , $_2$ $\frac{26s^2+36s+34}{18s^2 \;\; 24s \;\; 21}$ for s > ' .

Theorem

Let C 2 K^n be Minkowski centered with s(C) = s, n even. Then

(i) $C \setminus (\ \ C) \quad \frac{n}{n+1}\mathrm{conv}(C [\ (\ \ C))$, if s $\quad {}_2(n)$, and

(ii) $\frac{C \ + (\ \ C)}{2} \qquad \frac{n(n+2)}{(n+1)^2} \frac{C \quad C}{2}$, if s $\quad {}_3(n)$, where

$$:= \quad (n;s) := \frac{(n \quad s+1)(s+1)}{1 \quad (n \quad s)(n+s(n+1))} \quad n;$$

$$:= \quad (n;s) = (\ n+1) \quad 1 + \frac{sn}{s+1} \quad \frac{1+n \quad s}{1 \quad n(n \quad s)} \quad n \quad ;$$

$$_2 := \quad {}_2(n) := \frac{n^4+n^3+2n^2+ \text{p} \overline{\ \ {}_2}}{2(n^3+2n^2+3n+1)};$$

$$_2 := \quad {}_2(n) := n^8 + 6n^7 + 17n^6 + 28n^5 + 28n^4 + 12n^3 \quad 4n^2 \quad 12n \quad 4:$$

Idea of the proof

For $C; K \; 2 \; K^n$ the Banach-Mazur distance between K and C is

$$d_{BM}(K; C) := \inf f \quad 1 : c^1 + K \quad L(C) \quad c^2 + \quad K; L \; 2 \; GL(n) g:$$

Proposition (Schneider, 2009)

Let $S \; 2 \; K^n$ be an n-simplex and $C \; 2 \; K^n$ s.th. $s(C) = n$ $\quad$ ", " $2 \; (0; \frac{1}{n})$.
Then

$$d_{BM}(C; S) \quad 1 + \frac{(n+1)"}{1 \quad n"}:$$

For $s \; 2 \; (n \quad \frac{1}{n}; n)$: $c^1 + S \quad L(C) \quad c^2 + \quad S$ with $\quad \frac{n+1 \quad s}{1 \quad n(n \quad s)}$.

Theorem

Let C 2 K ^{n} be Minkowski centered, 2 R. Then

$$C \setminus (\quad C) \quad ^{opt} \quad conv(C [\ (\quad C)) \quad i$$

$$L(C) \setminus L(\quad C) \quad ^{opt} \quad conv(L(C) [\ L(\quad C)):$$

Thus, $c^1 + S \quad C \quad c^2 + \quad S$ with $\frac{n+1 \quad s}{1 \quad n(n \quad s)}$.

Show: 02 $c^1 + S \quad C \quad c^2 + \quad S$.

De ne: as minimal distance from 0 to the facets of $c^1 + S$.

Show: $\frac{n+1}{s+1}(1 \quad s(\quad 1)) =: \quad 1$.

Combine:

$$C \backslash (\ \ C) \quad (c^2 + \ \ S) \backslash (\ \ c^2 \quad S)$$

$$(\ \ + n(\quad))(S \backslash (\ \ S))$$

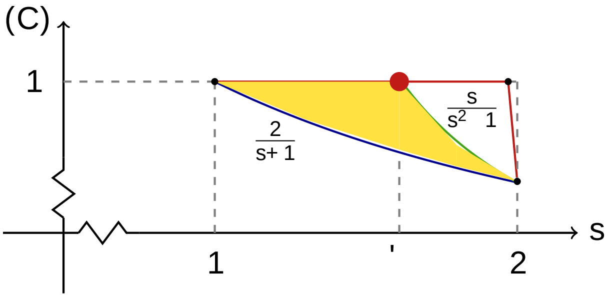
$$\frac{n}{n+1}(\ \ + n(\quad))\mathrm{conv}(S [\ (\ \ S))$$

$$\frac{n}{n+1}\frac{(\ \ + n(\quad))}{}\mathrm{conv}((c^1 + S) [\ (\ \ c^1 \quad S))$$

$$\frac{n}{n+1}\frac{(\ \ + n(\quad))}{}\mathrm{conv}(C [\ (\ \ C)):$$

Let C 2 K ^{n}, (C) > 0 s.th. C \ (C) ^{opt} (C) conv(C [(C)).
For s 2 [1; n] we de ne

$${}_{1n2}(s) := \inf \, n \sup f \quad (C) : C\ 2\ K^n \text{ Minkowski centered}; s(C) = sg:$$

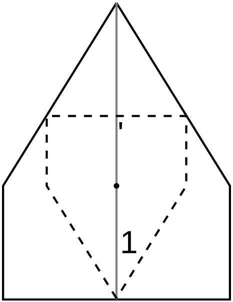


For n = 2: $_1$(s) = $\frac{2}{s+1}$. $_2$(s) = 1 for s ' := $\frac{1+ \text{p} \overline{5}}{2}$;
$_2$(s) $\frac{s}{s^2 \quad 1}$ for s ' , $_2$ $\frac{26s^2 + 36s + 34}{18s^2 \quad 24s \quad 21}$ for s > ' .

Golden House

Theorem

Let C 2 K ^{2} be Minkowski centered, $\frac{C \quad C}{2}$ opt $\frac{C \quad C}{2}$, then s(C) ' .
If s(C) = ' , 9 linear transformation L s.th. L(C) = golden house.



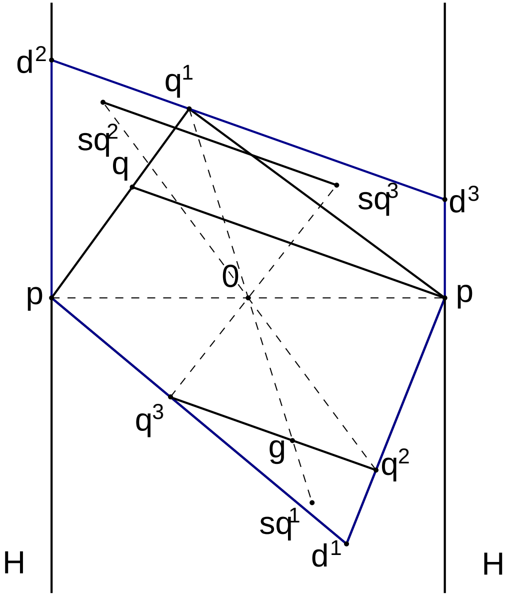
Idea of the proof

Proposition (Brandenberg, Koenig, 2013)

Let $K, C \in \mathcal{K}^n$ and $K \subset C$. The following are equivalent:

(i) $K \subset^{opt} C$.

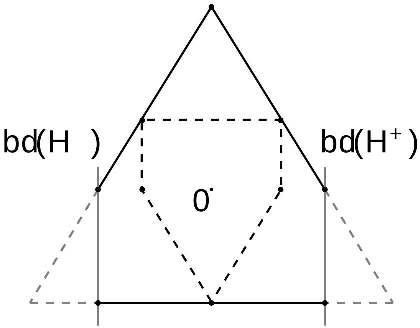
(ii) There exist $k \in \{2, \ldots, n+1\}$, $p^j \in K \cap \mathrm{bd}(C)$, $a^j \in N(C, p^j)$, $j = 1, \ldots, k$, such that $0 \in \mathrm{conv}(\{a^1, \ldots, a^k\})$.


d2
q1
sq2
q
sq3
d3
0
p
p
q3
g
q2
sq1
d1
H
H

Generalized Golden House

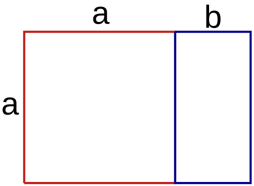
Let n 2. Then for C = generalized golden house holds

$$C \setminus (\quad C) \quad {}^{opt} \; \mathrm{conv}(C [\; (\quad C)); \quad s(C) = \frac{1}{2}(n \quad 1 + \sqrt{(n \quad 2)n + 5})$$



Golden Ratio

$$\frac{a+b}{a} = \frac{a}{b} =: \varphi \quad \Rightarrow \quad \varphi^2 - \varphi - 1 = 0 \quad \Rightarrow \quad \varphi = \frac{\sqrt{5}+1}{2}.$$



Generalized Golden Ratio

$$\frac{(n-1)a+b}{a} = \frac{a}{b} =: \varphi \implies \varphi^2 - (n-1)\varphi - 1 = 0$$
$$\implies \varphi = \frac{1}{2}\left(n-1+\sqrt{(n-2)n+5}\right).$$