EXCERCIZE SET 1, FOLLOW UP TO PUTNAM 2015

1. Find all the complex roots for:

a) $z^5 = 1$;

b) $z^{12} = 1$.

2. Recall that for a positive integer n, the Euler's totient function is defined as:

$$\phi(n) = \#\{k \in \{1, ..., n-1\} : gcd(n, k) = 1\}.$$

a) Prove that $\phi(p) = p - 1$, given a prime number p;

b) Prove that $\phi(p^k) = p^{k-1}(1 - p)$;

c) Prove that $\phi(mn) = \phi(m)\phi(n)$;

d) Compute $\phi(30)$; $\phi(242)$; $\phi(210)$; $\phi(60)$.

3. Let $A(1, 1)$ and $B(3, 9)$. Find the area of the region between the chord AB and the graph of the function $y = x^2$.

4. a) Let $a_0 = 1$, $a_1 = 3$, $a_n = 4a_{n-1} - 3a_{n-2}$. Find an expression for a_n in terms of n.

b) Let $a_0 = 1$, $a_1 = 3$, $a_n = 8a_{n-1} - a_{n-2}$. Find an expression for a_n in terms of n.