Rigidity results of Alexandrov type

based on joint works with DORIN BUCUR (U. Savoie)

1. Alexandrov Thm - moving planes
2. Link with the isoperimetric inequality - rigidity for sets with finite perimeter
3. Interplay with PDE's
4. Rigidity for K-dense domains

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5. A new rigidity problem.
6. Link with Riesz rearrangement inequality
7. Fractional rigidity
8. Rigidity for r-critical sets - new moving planes

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9. Rigidity for general kernels
10. Rigidity for polygons.

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1. Alexandrov Thm (1958)

Thm.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded connected domain, with $\partial\Omega \in C^2$.

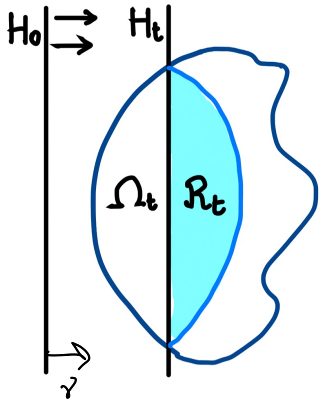
Then

$$H_\Omega(x) = c > 0 \quad \forall x \in \partial\Omega \iff \Omega \text{ is a ball}$$

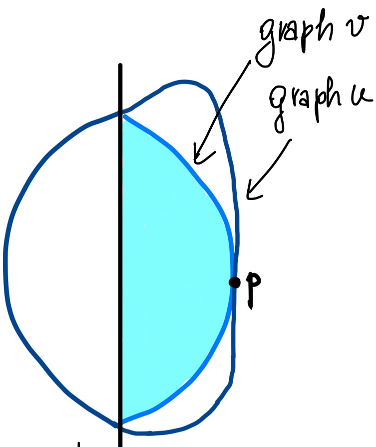
- false if Ω is not bounded (Ω halfspace)
- false if Ω is not connected ($\Omega = B_1 \sqcup B_2$)
- ??? if Ω is not C^2 (H_Ω ???)
- false if $\partial\Omega \rightsquigarrow S$ not embedded (Wente torus)

Proof. By moving planes

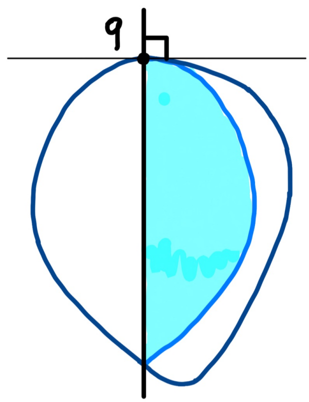
Start



Stop.



1) Interior tangency



2) Orthogonality

Goal: $w := u - v$, $\boxed{w \equiv 0 \text{ near } p/q}$

We know: $w(p) = 0$, $w \le 0$ near p.

$$H_\Omega = \frac{1}{n-1}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \frac{1}{n-1}\left(\frac{\Delta u}{\sqrt{1+|\nabla u|^2}} - \frac{\nabla^2 u \cdot \nabla u \cdot \nabla u}{(1+|\nabla u|^2)^{3/2}}\right)$$

$$\Downarrow$$

$$\sum_{i,j} a_{ij}(x)\,\partial^2_{ij} w + \sum_k b_k(x)\,\partial_k w = 0 \qquad \forall x \in \mathcal{U}(p/q)$$

uniformly elliptic with bounded coefficients

1) <u>Interior tangency</u> : w takes its max at an interior point

$\Rightarrow$ by the STRONG MAX PRINCIPLE, $w \equiv 0$ near p

2) <u>Orthogonality</u> : w takes its max at a boundary point

$\Rightarrow$ by HOPF BOUNDARY POINT LEMMA

- $\frac{\partial w}{\partial \nu}(q) < 0$ $\leftarrow$ ruled out by orthogonality
- $w \equiv 0$ near q

<u>Conclusion</u> : Ω has a plane of symmetry in every direction.

$\boxtimes$

2. Link with the isoperimetric inequality - rigidity for sets with finite perimeter

Let Ω be as in Alexandrov Thm.

$$\mathcal{Q}(\Omega) := \frac{Per(\Omega)}{|\Omega|^{\frac{n-1}{n}}}$$ isoperimetric quotient

Def. Ω is a critical set for $\mathcal{Q}$ if

$$\frac{d}{dt}\mathcal{Q}(\Omega_t)\Big|_{t=0} = 0 \quad \forall\ \Omega_t = \phi_t(\Omega) \text{ perturbations of } \Omega$$
$$\phi_t(x) = x + tX(x) + o(t), \quad X \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$$

__Remark__: Ω is a critical set for $\mathcal{Q}$ $\Longleftrightarrow$ $H_\Omega(x) = c \ \forall x \in \partial\Omega$

remains true for sets of finite perimeter

- $$\frac{d}{dt}|\Omega_t|\Big|_{t=0} = \int_\Omega \operatorname{div} X \overset{1}{=} \int_{\partial^*\Omega} X\cdot\nu_\Omega \, d\mathcal{H}^{n-1} \quad \forall X \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$$
- $$\frac{d}{dt}Per(\Omega_t)\Big|_{t=0} = \int_{\partial\Omega} \underbrace{\operatorname{div}_{\partial\Omega}}_{\operatorname{div}X - \nu_\Omega\cdot\nabla X[\nu_\Omega]} X \overset{2}{=} \int_{\partial\Omega} \underset{L^1_{loc}(\partial^*\Omega;d\mathcal{H}^{n-1})}{H_\Omega}\, X\cdot\nu_\Omega \, d\mathcal{H}^{n-1}$$
$$\forall X \in C^1_c(\mathbb{R}^n;\mathbb{R}^n).$$

__Question__: Does Alexandrov Thm hold for sets with finite perimeter?

Thm. [Delgadino-Maggi, Anal & PDE's 2019]

Among sets with finite vol and finite per, the unique critical sets for $\mathcal{Q}$ (or sets with constant distributional mean curvature) are finite unions of equal balls.

Idea of the proof

HEINTZE-KARCHER INEQUALITY: $\Omega \in C^2$ with $H_\Omega > 0$

$$|\Omega| \leq \frac{n-1}{n} \int_{\partial\Omega} \frac{1}{H_\Omega} \, d\mathcal{H}^{n-1}, \quad \text{with } = \iff \Omega = B$$

$$H_\Omega = c \text{ on } \partial\Omega \Rightarrow \underset{(n-1)\,\mathrm{Per}(\Omega)}{\underset{\parallel}{\int_{\partial\Omega} H_\Omega \, x\cdot\nu}} = \underset{c\, n|\Omega|}{\underset{\parallel}{\int_{\partial\Omega} c \, x\cdot\nu}} \Rightarrow c = \frac{n-1}{n} \frac{\mathrm{Per}(\Omega)}{|\Omega|}$$

$\blacksquare$

3. Interplay with PDE's

- Elliptic PDE's

Thm. [Serrin, ARMA 1971]

Consider the overdetermined problem:

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ |\nabla u| = c & \text{on } \partial\Omega \end{cases}$$

$\exists$ solution $\iff$ $\Omega = B$

- Parabolic PDE's

Thm. [Magnanini - Sakaguchi, Ann. of Math 2002]

Consider Matzoh ball soup problem:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega \times (0,+\infty) \\ u = 0 & \text{in } \Omega \times \{0\} \\ u = 1 & \text{on } \partial\Omega \times (0,+\infty) \end{cases}$$

the isothermic surfaces $\{u(\cdot,t) = \text{const}\}$ are stationary

$\iff$ $\Omega = B$

Thm. [Magnanini - Prajapat - Sakaguchi, TAMS 2006]

Consider the problem:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times [0,+\infty) \\ u(x,0) = \chi_\Omega(x) & \text{in } \mathbb{R}^n \end{cases}$$

$\partial\Omega$ is stationary isothermic $\iff$ Ω is B-dense

$(u(x,t) = \alpha(t) \quad \forall x \in \partial\Omega, \ \forall t \in (0,+\infty))$

Def. Ω is B-dense $\iff$ $\forall r > 0 \ \exists \ c = c(r)$ such that

$$|\Omega \cap B_r(x)| = c(r) \quad \forall x \in \partial\Omega$$

Question: Let Ω be bounded (measurable)

$$\Omega \text{ is B dense} \overset{???}{\iff} \Omega = B$$

as $r \to 0^+$

- $|\Omega \cap B_r(x)| = \frac{1}{2}\omega_n r^n - \gamma_n H_\Omega(x) r^{n+1} + o(r^{n+1})$

[Hulin - Troyanov, Amer. Math. Monthly 2003]

4. Rigidity for K-dense domains

Let Ω be a set of finite Lebesgue measure, let K be a convex body with $0 \in \mathrm{int}(K)$

Def. Ω is K-dense if $|\Omega \cap (x+rK)| = c(r)$ $\forall x \in \partial\Omega$, $\forall r > 0$

Question: what can be said about Ω (and K)?

Thm. [Magnanini–Marini, Proc. Roy Soc Edinburgh 2016]
Ω is K-dense $\Leftrightarrow$ Ω and K are homothetic ellipsoids
(In particular, if K is a ball, Ω is a ball).

Idea of the proof.

$$|\Omega \cap (x+rK)| = |\Omega| + W(x)\,(r_{\Omega,K}(x) - r)^{\frac{n+1}{2}} + o(\dots)$$

as $r \to r_{\Omega,K}(x) = \inf\{r>0 : \Omega \subseteq x + rK\}$

$$W(x) = \text{const} \iff \begin{cases} \Omega \text{ and } K \text{ are homothetic} \\ G_K = c\, h_K^{n+1} \Rightarrow K \text{ is an ellipsoid.} \end{cases}$$

Petty Thm.

G_K: Gaussian curvature; h_K: support function

∎

5. A new rigidity problem.

Fix $r>0$.

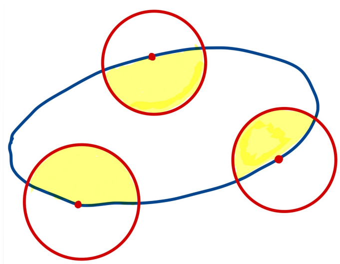
Characterize measurable sets $\Omega \subseteq \mathbb{R}^n$, with $|\Omega|<+\infty$, such that

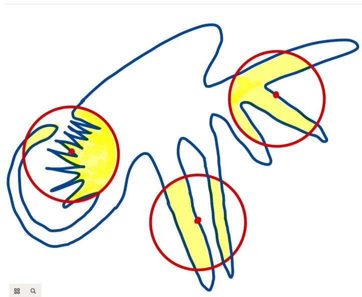
$$\boxed{|\Omega \cap B_r(x)| = c \qquad \forall x \in \partial^*\Omega} \quad \leftarrow \Omega \text{ is "r-critical"}$$

$$\left(\partial^*\Omega := \left\{ x \in \mathbb{R}^n : \limsup_{\rho\to 0} \frac{|\Omega \cap B_\rho(x)|}{\rho^n} > 0, \ \limsup_{\rho \to 0} \frac{|\Omega^c \cap B_\rho(x)|}{\rho^n} > 0 \right\}\right)$$

$\cap\!|$

$\partial\Omega$





Historical note (on the origins of the problem)

- [Cimmino, Rend. Circ. Mat. Palermo 1932]:

Is it possible to characterize smooth surfaces $\Gamma = \partial\Omega$ in $\mathbb{R}^3$ such that

$$\mathcal{H}^2(\Omega \cap \partial B_r(x)) = 2\pi r^2 \quad \forall x \in \Gamma, \ \forall r > 0 \text{ suff. small}$$

- [Nitsche, Analysis 1995]

The only smooth surfaces with this property are: the plane and the right helicoid

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More references (on the noncompact case)

- [Meeks-Rosenberg, Ann. of Math. 2005]

The plane and the right helicoid are the unique simply connected minimal surfaces embedded in $\mathbb{R}^3$

- [Kapouleas, Ann. of Math 1990]

A general construction of CMC surfaces in $\mathbb{R}^3$ (many noncompact examples are embedded!)

Back to our problem:

Characterize measurable sets $\Omega \subseteq \mathbb{R}^n$, with $|\Omega| < +\infty$, such that

$$\boxed{|\Omega \cap B_r(x)| = c \qquad \forall x \in \partial^* \Omega} \leftarrow \Omega \text{ is "r-critical"}$$

Generalized version:

$$|\Omega \cap B_r(x)| = \int_\Omega \chi_{B_r(x)}(y)\, dy = \int_\Omega \chi_{B_r(0)}(x-y)\, dy$$

$\chi_{B_r(0)} \rightsquigarrow h : \mathbb{R}^n \to \mathbb{R}_+$ radially symmetric

Characterize measurable sets $\Omega \subseteq \mathbb{R}^n$ such that

$$\boxed{\int_\Omega h(x-y)\, dy = c \qquad \forall x \in \partial^* \Omega} \leftarrow \Omega \text{ is "h-critical"}$$

6. Link with Riesz rearrangement inequality

- Given $h : \mathbb{R}^n \to \mathbb{R}_+$ radially symmetric ↘

$$\underbrace{\int_\Omega \int_\Omega h(x-y)\,dx\,dy}_{\overset{\shortparallel}{J_h(\Omega)}} \leq \int_{\Omega^*}\int_{\Omega^*} h(x-y)\,dx\,dy$$

Ω^*: ball with $|\Omega^*| = |\Omega|$

- Ω h-critical $\iff \left.\frac{d}{dt} J_h(\Omega_t)\right|_{t=0} = 0 \quad \forall \Omega_t = \phi_t(\Omega)$ volume preserving

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Nonlocal perspective:

- $$J_h(\Omega) = c - \underbrace{\int_\Omega \int_{\Omega^c} h(x-y)\,dx\,dy}_{\overset{\shortparallel}{h\text{-Per}(\Omega)}}$$

- Riesz inequality $\rightsquigarrow$ $h\text{-Per}(\Omega) \geq h\text{-Per}(\Omega^*)$

- $\int_\Omega h(x-y)\,dy$ $\rightsquigarrow$ nonlocal h-mean curvature

$$\left(\int_{\mathbb{R}^n} h(x-y)\,[\chi_{\Omega^c}(y) - \chi_\Omega(y)]\,dy \right)$$

[Bourgain - Brézis - Mironescu, Optimal Control & PDEs, 2001]

[Mazón - Rossi - Toledo, Birkhäuser 2019]

7. Fractional rigidity $\left(h(x)=\frac{1}{|x|^{n+2s}},\ s\in(0,\frac{1}{2})\right)$

$$\mathrm{Per}_s(\Omega)=\int_\Omega\int_{\Omega^c}\frac{1}{|x-y|^{n+2s}}\,dx\,dy$$

$(\partial\Omega\in C^{1,\alpha},\ \alpha>2s)$

$$H_s(\Omega)=\int_{\mathbb{R}^n}\frac{\chi_{\Omega^c}(y)-\chi_\Omega(y)}{|x-y|^{n+2s}}\,dy$$

[Caffarelli - Souganidis, CPAM 2008]
[Caffarelli - Roquejoffre - Savin, CPAM 2010]

- Fractional isoperimetric inequality:

$\mathrm{Per}_s(\Omega)\geq\mathrm{Per}_s(\Omega^*)$ [Frank-Seiringer, JFA 2008]

- Fractional Alexandrov Theorem:

Ω bounded open set of class $C^{1,\alpha}$ with $H_s=c$ on $\partial\Omega\Rightarrow\Omega=B$

[Cabré - Fall - Solà Morales - Weth, J. Reine Angew. Math. 2018]
[Ciraolo - Figalli - Maggi - Novaga, " " " "]

→ no bubbling!

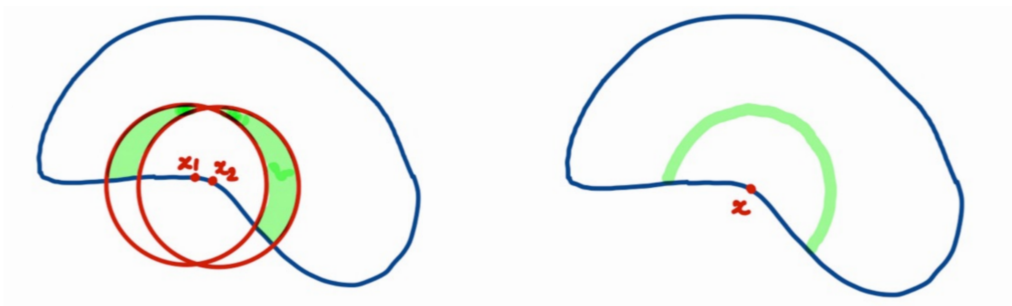
→ proof by moving planes.

8. Rigidity for r-critical sets

The kernel $h = \chi_{B_r(0)}$ is:
- BOUNDED
- COMPACTLY SUPPORTED
- DISCONTINUOUS

Def. A measurable set Ω is r-nondegenerate if

$$\inf_{x_1, x_2 \in \partial^* \Omega} \frac{|\Omega \cap (B_r(x_1) \Delta B_r(x_2))|}{|x_1 - x_2|} > 0$$



Lemma:

Ω open connected $\Rightarrow$ Ω r-nondegenerate $\forall r < \operatorname{diam} \Omega$

- same for Ω of finite perimeter indecomposable
- if $\Omega = \bigsqcup_i \Omega_i$ $\rightsquigarrow$ $r < \inf_i (\operatorname{diam} \Omega_i)$

components of Ω

Thm. [Bueno - F. , ArXiv Preprint 2021]

Let $r>0$, and let $\Omega \subseteq \mathbb{R}^n$ be measurable with $|\Omega|<+\infty$.
Assume Ω is r-critical and r-nondegenerate.
Then Ω is equivalent to a finite union of equal balls, of radius $R > \frac{r}{2}$, at mutual distance larger than or equal to r.

"Regular" cases:

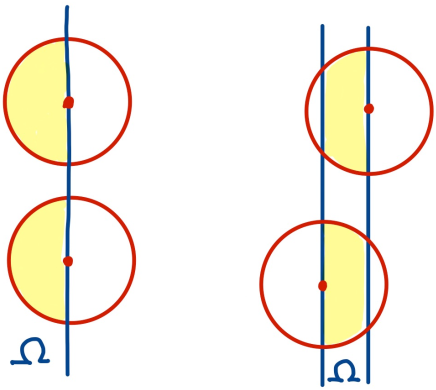
- Ω open connected set, with $|\Omega|<+\infty$
  Ω r-critical, $r < \operatorname{diam} \Omega \Rightarrow \Omega = B$.

- Ω of finite perimeter indecomposable, with $|\Omega|<+\infty$
  Ω r-critical, $r < \operatorname{diam} \Omega \Rightarrow \Omega = B$.

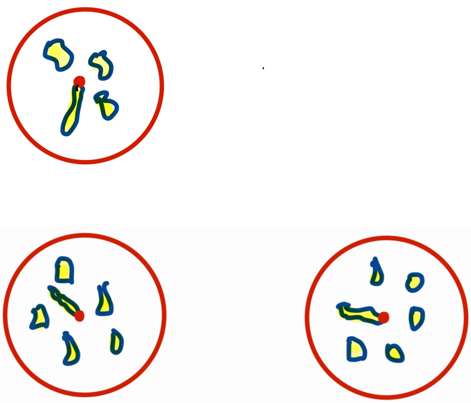
Remark: The same result holds if $B \rightsquigarrow E$ ellipsoid

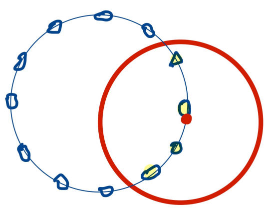
Sets escaping from rigidity (though r-critical)

- Sets of infinite measure



- r-degenerate sets

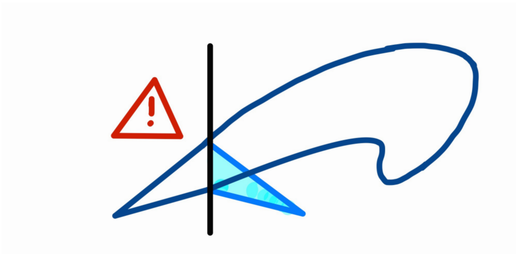




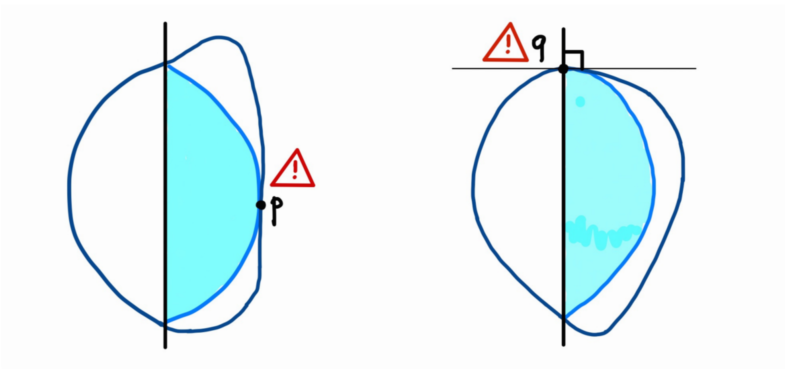
Proof.

Classical moving planes fail

- Start:



- Stop:



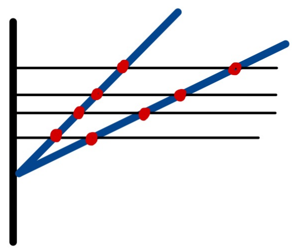
- Conclusion:

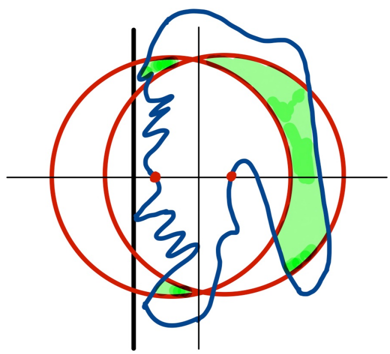
New moving planes

- Start: Under the hypotheses of the Thm, for $t \ll 1$

$R_t \subseteq \Omega$ and $\Omega_t \cup R_t$ is Steiner symmetric about H_t

(By contradiction)

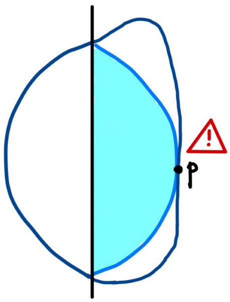


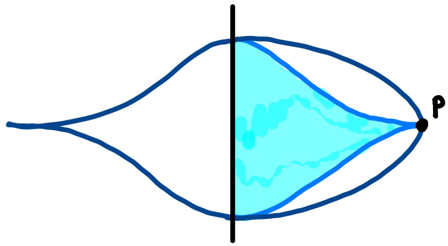


- <u>Stop</u>:

1) Interior tangency ⇝ Away contact

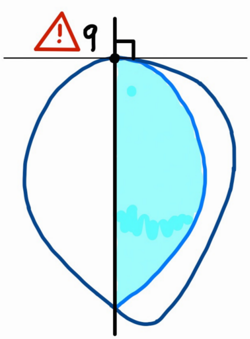
$$p \in \left(\overline{\partial^*\Omega} \cap \overline{\partial^* R_t}\right) \setminus H_t$$

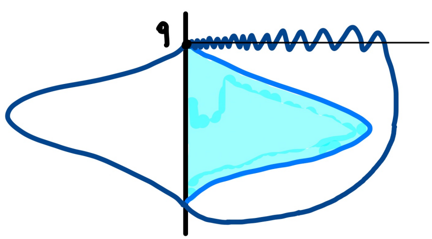




2) Orthogonality ⇝ Close contact

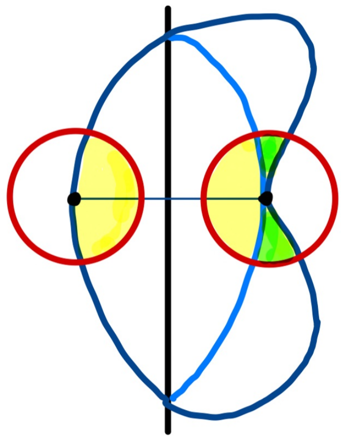
$$\underset{\substack{\cap \\ H_t}}{q} = \lim_n \underset{\substack{\cap \\ \partial^*\Omega}}{q_n^1} = \lim_n \underset{\substack{\cap \\ \partial^*\Omega}}{q_n^2}, \quad \Pi_{H_t}(q_n^1) = \Pi_{H_t}(q_n^2)$$





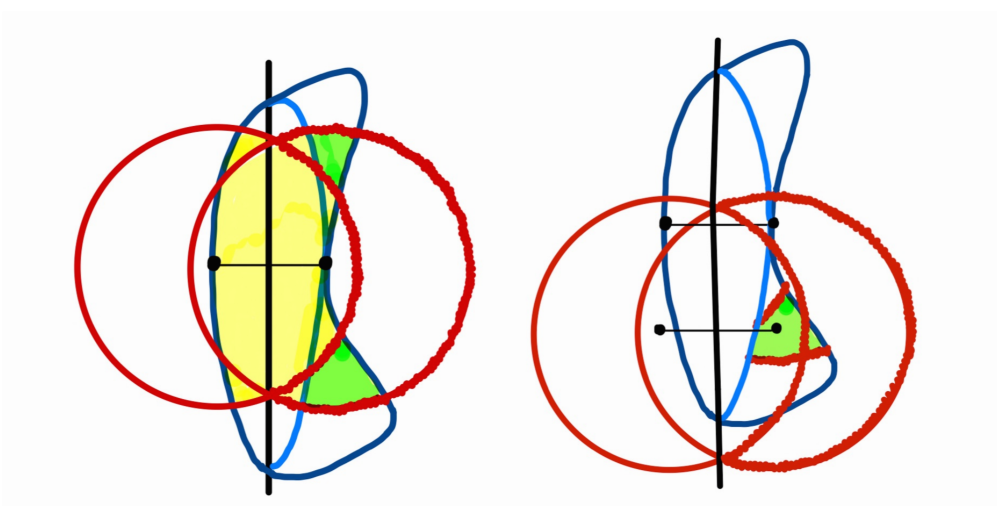
1) Away contact $\Rightarrow$ local symmetry

Easy situation: $B_r(p) \cap B_r(p') = \phi$ (use r-criticality)



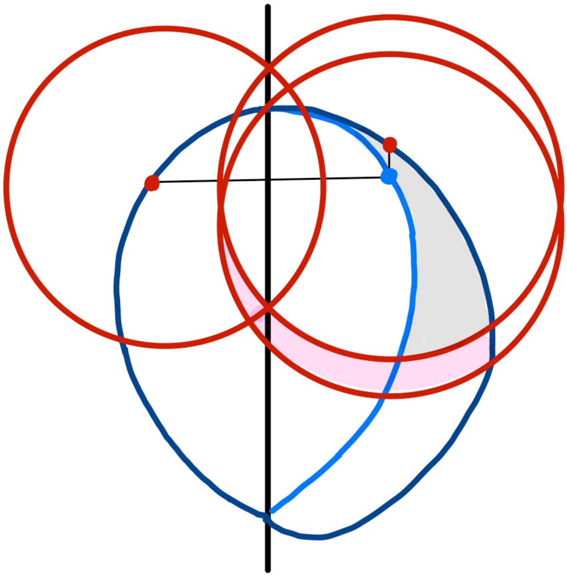
More delicate situation: $B_r(p) \cap B_r(p') \neq \phi$

use symmetry inside r-moons and a ping-pong game



2) Close contact: not possible without away contact

(Based on a local analysis exploiting nondegeneracy)

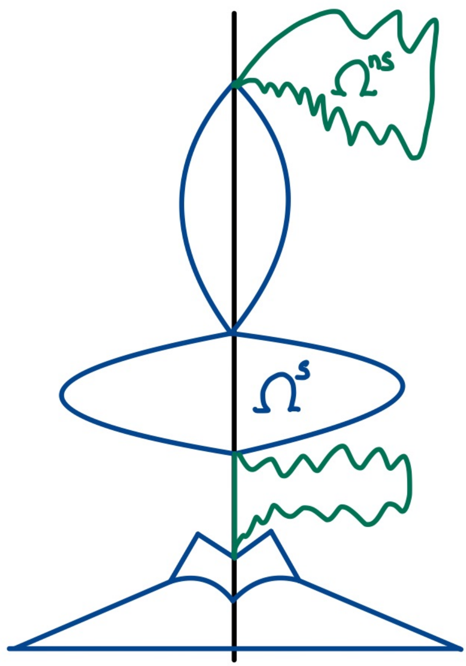


- Conclusion

→ Under some "connectedness" hyp $\Rightarrow \Omega = B$

→ In general: $\Omega = \Omega^s \sqcup \Omega^{ns}$, with Ω^s OPEN $\Rightarrow$

$$\begin{cases} \Omega^s = \text{finite union of balls} \\ \Omega^{ns} \text{ Lebesgue negligible} \end{cases}$$



9. Rigidity for general Kernels

Let $h \in L^1_{loc}(\mathbb{R}^n; \mathbb{R}_+)$ radially symmetric nonincreasing.
Let $\Omega \subseteq \mathbb{R}^n$ be measurable, with $|\Omega| < +\infty$.

$$|\Omega \cap B_r(x)| = c \quad \forall x \in \partial^*\Omega$$

$$\Downarrow$$

$$\int_\Omega h(x-y)\,dy = c \quad \forall x \in \partial^*\Omega$$

Ω is h-critical

$$\inf_{x_1, x_2 \in \partial^*\Omega} \frac{|\Omega \cap (B_r(x_1) \Delta B_r(x_2))|}{|x_1 - x_2|} > 0$$

$$\Downarrow$$

$$\inf_{x_1, x_2 \in \partial^*\Omega} \int_\Omega \frac{|h(x_1-y) - h(x_2-y)|\,dy}{|x_1 - x_2|} > 0$$

Ω is h-nondegenerate

Remark:

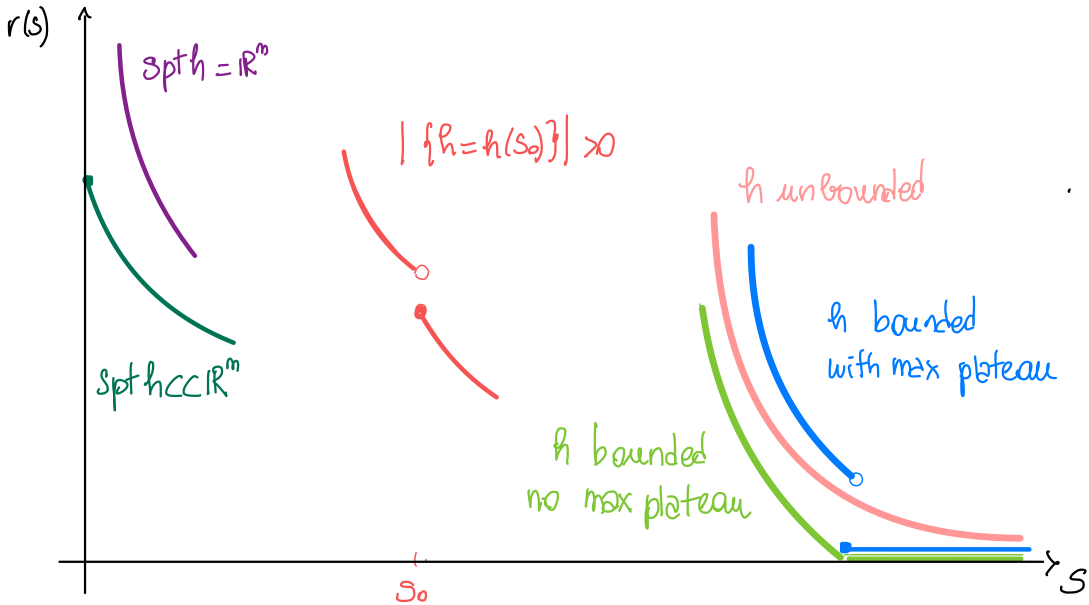
$$\int_\Omega h(x-y)\,dy = \int_0^{+\infty} |\Omega \cap B_{r(s)}(x)|\,ds, \quad \text{where}$$

$$\boxed{B_{r(s)}(0) = \{h > s\}}$$

If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $\varphi(x) := h(|x|)$

$r(s) = \mathcal{L}^1(\{x \in \mathbb{R}_+ : \varphi(x) > s\})$ DISTRIBUTION FUNCTION OF φ

Properties of the map $s \mapsto r(s) = \mathcal{L}^1(\{\varphi > s\})$



- nonincreasing, with $\sup_{(0,+\infty)} r(s) = r(0^+) = \mathcal{L}^1(\{\mathrm{spt}\,\varphi\})$
- right continuous

  continuous at $s = s_0 \iff \mathcal{L}^1(\{\varphi = \varphi(s_0)\}) = 0$
- $r(s) = 0 \quad \forall s \geq \operatorname{ess\,sup} \varphi$
- $\eta := \mathcal{L}^1(\{\varphi = \operatorname{ess\,sup} \varphi\})$
- $\sigma := \begin{cases} \sup\{s : \ r(s) > \operatorname{diam}\Omega\} & \text{if } r(0) > \operatorname{diam}\Omega \\ 0 & \text{if } r(0) \leq \operatorname{diam}\Omega \end{cases}$

$$r(\sigma) < \operatorname{diam}\Omega \iff \begin{cases} \mathcal{L}^1(\{\varphi = \varphi(\operatorname{diam}\Omega)\}) > 0 \\ r(0) < \operatorname{diam}\Omega \end{cases}$$

Theorem [Bucur - F., ArXiv Preprint 2022]

Let $\Omega \subseteq \mathbb{R}^n$ with $|\Omega| < +\infty$ be h-critical and h-nondegenerate

Assume that $\displaystyle\int_1^{+\infty} r(s)^{n-1}\, ds < +\infty \qquad (*)$

Then Ω is a finite union of balls B_i of the same radius R.

Moreover: $R > \frac{\eta}{2}$ and $\mathrm{dist}(B_i, B_j) \geq r(\sigma)$

On the assumptions

- About condition $(*)$ (improved integrability)

$$\int_{K \atop \cap \mathbb{R}^n} h = \int_0^{+\infty} |K \cap B_{r(s)}|\, ds \Rightarrow h \in L^1_{loc} \text{ provided } \int_1^{+\infty} r(s)^n\, ds < +\infty$$

- Lemma (about nondegeneracy)

Ω open connected, $r < \mathrm{diam}\,\Omega \Rightarrow \Omega$ r-nondegenerate

$\rightsquigarrow$ Ω open connected, $\eta < \mathrm{diam}\,\Omega \Rightarrow \Omega$ h-nondegenerate

On the geometric impact of the kernel

- R is bounded from below $\Leftrightarrow \eta > 0$ (maximal plateau)

- Multiple balls are allowed $\Leftrightarrow r(\sigma) < \mathrm{diam}\,\Omega$

Example 1

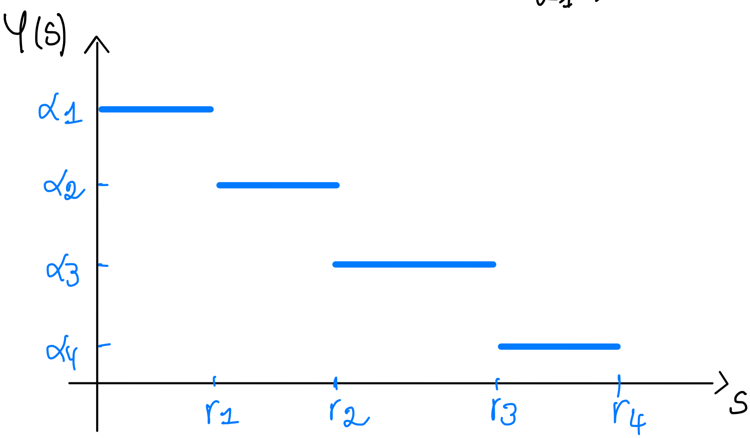
$$h(x) = \frac{1}{|x|^\alpha}$$

improved integrability OK for $\alpha < n-1$

- $\eta = 0 \Rightarrow R > 0$
- $r(\sigma) = +\infty \Rightarrow$ multiple balls not allowed.

Example 2

$$h(x) = \sum_{i=1}^{N} \alpha_i \chi_{B_{r_i}(0) \setminus B_{r_{i-1}}(0)} \qquad \alpha_1 > \dots > \alpha_N \qquad r_1 < \dots < r_N$$



improved integrability OK (since h is bounded)

- $\eta = r_1 \Rightarrow R > \frac{r_1}{2}$
- $r(\sigma) \in \{r_1, \dots, r_N\} \not\Rightarrow$ multiple balls allowed

On the relationship with Burchard's work

[Burchard, Ann of Math 1996]

- Characterization of equality cases in general Riesz inequality:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(y)h(x-y)\,dx\,dy \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^*(x)g^*(y)h^*(x-y)\,dx\,dy$$

$$\left(f=g=\chi_\Omega,\ h=h^* \rightsquigarrow \int_\Omega\int_\Omega h(x-y)\,dx\,dy \leq \int_{\Omega^*}\int_{\Omega^*} h(x-y)\,dx\,dy\right)$$

- Case of characteristic functions: $f=\chi_{\Omega_1}, g=\chi_{\Omega_2}, h=\chi_{\Omega_3}$

  $\Rightarrow \Omega_i$ are balls (homothetic ellipsoids) or homothetic convex bodies

  In particular: if Ω_3 is a ball $\Rightarrow \Omega_1, \Omega_2$ are balls

- Case of arbitrary functions:

  Almost all level sets must produce equality (hard to check!)

  In particular: if $f=g=\chi_\Omega$,

  only for h STRICTLY DECREASING, it follows that Ω is a ball

$\Rightarrow$ for h non strictly decreasing, our result gives some NEW INFORMATION about possible maximizers (CRITICAL SETS)

even though the ADDITIONAL HYPOTHESES of improved integrability and h-nondegeneracy.

10. Rigidity for polygons

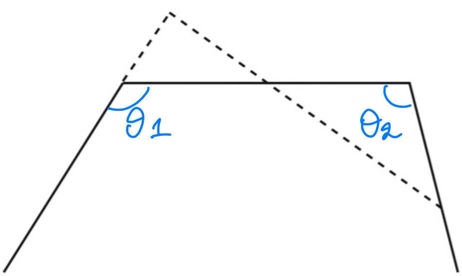
- <u>Polygonal isoperimetric inequality</u> $\qquad Q(P) = \dfrac{|\partial P|}{|P|^{1/2}}$

$$Q(P) \geq Q(P_N^*) \qquad \forall P \in \mathcal{P}_N = \{N\text{-gons}\}$$

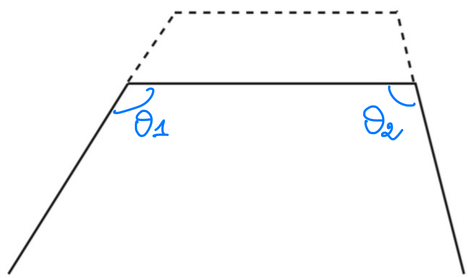
$\uparrow$ regular N-gon

Moreover, P_N^* is the unique <u>critical</u> N-gon, namely the unique N-gon such that

$$\frac{d}{dt} Q(P_t)\Big|_{t=0} = 0 \qquad \text{for the following perturbations } \{P_t\}:$$



ROTATION AROUND MID-POINT



PARALLEL MOVEMENT

$$\begin{cases} \dfrac{d}{dt} |\partial P_t|\Big|_{t=0} = \dfrac{1}{2}\left(f(\theta_1) - f(\theta_2)\right) \\ \dfrac{d}{dt} |P_t|\Big|_{t=0} = 0 \end{cases}$$

$$\begin{cases} \dfrac{d}{dt} |\partial P_t|\Big|_{t=0} = f(\theta_1) + f(\theta_2) \\ \dfrac{d}{dt} |P_t|\Big|_{t=0} = \ell \end{cases}$$

$$\left(f(\theta) = \cot\theta + \frac{1}{\sin\theta} \right)$$

Question : what happens in the polygonal setting when dealing with Riesz type energies (or nonlocal perimeters)?

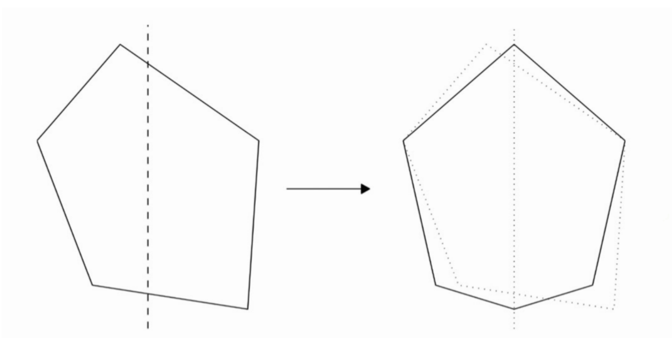
$$\mathcal{J}_h(P) = \int_P\int_P h(x-y)\,dx\,dy \qquad P \in \mathcal{P}_N = \{N\text{-gons}\}$$

① Is P_N^* a maximizer under area constraint?

$$\mathcal{J}_h(P) \overset{???}{\leq} \mathcal{J}_h(P_N^*) \quad \forall P \in \mathcal{P}_N = \{N\text{-gons}\},\ |P| = |P_N^*|$$

CURRENTLY OPEN, EXCEPT FOR $N=3,4$ (true)

the proof is based on Steiner symmetrization



$N \geq 5 \Rightarrow$ proof fails!

[Bonacini - Cristoferi - Topaloglu , JGA 2022]

[Polya-Szegö , Isop. inequalities in Math. Physics 1951]

[Brascamp - Lieb - Luttinger , JFA 1974]

(2) Is P_N^* critical for J_h under area constraint? YES

Is it unique ??? ($\rightarrow$ answer to question (1)!)

CURRENTLY OPEN, EXCEPT FOR $N=3,4$ (true)

Setting $\quad v_P(x) = \int_P h(x-y)\,dy$,

the stationariety conditions read:

ROTATION AROUND MID-POINT

$$(*) \int_{\overline{P_i M_i}} v_P(x)\,|x-M_i|\,d\mathcal{H}^1 = \int_{\overline{P_{i+1} M_i}} v_P(x)\,|x-M_i|\,d\mathcal{H}^1 \qquad \forall i=1,\dots,N$$

M_i mid-point of $\overline{P_i P_{i+1}}$

PARALLEL MOVEMENT

$$(**) \int_{\overline{P_i P_{i+1}}} v_P(x)\,d\mathcal{H}^1 = \underset{\substack{\uparrow\\ \text{independent of } i}}{c}\;\mathcal{H}^1(\overline{P_i P_{i+1}}) \qquad \forall i=1,\dots,N$$

Thm [Bonacini - Cristoferi - Topaloglu, JGA 2022]

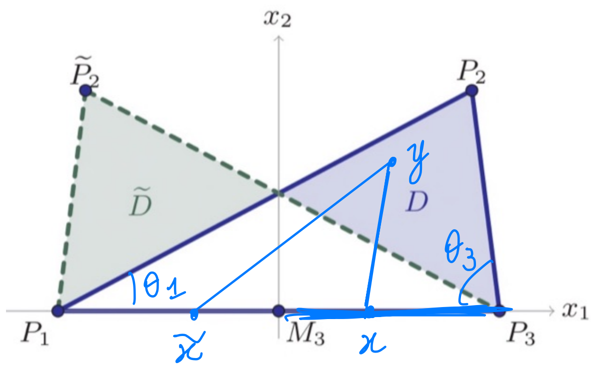
The regular triangle / square is the unique triangle / quadrilateral satisfying $(*)-(**)$.

(under some regularity hypotheses on the kernel)

Proof by reflection ($N=3$)

cf. [F. - Velichkov, Trans AMS 2019]

By contradiction



$$v_{\tilde{P}}(x) - v_P(x) = \int_{\tilde{P}} h(x-y)\,dy - \int_P h(x-y)\,dy$$

$$= \int_{\tilde{D}} h(x-y)\,dy - \int_D h(x-y)\,dy$$

$$= \int_D \left[h(\tilde{x}-y) - h(x-y)\right] dy < 0$$

$\uparrow$ h strictly decreasing

$$\Downarrow$$

$$\int_{M_3P_3} v_{\tilde{P}}(x)\,|x-M_3| < \int_{M_3P_3} v_P(x)\,|x-M_3|$$

$$\int_{P_1M_3} v_P(x)\,|x-M_3|$$

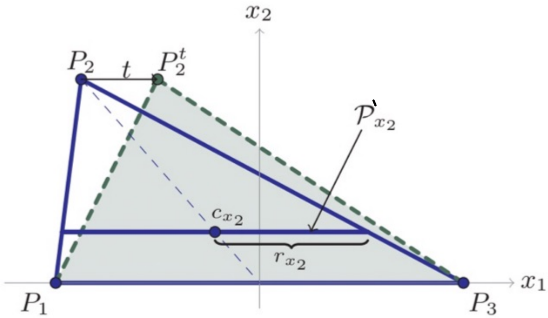
AGAINST (*)

$\square$

Proof by slices ($N=3$)

cf. [Carrillo-Hittmeier-Volzone-Yao, Inventiones 2019]

By contradiction:



$$J_h(P) = \int_P\int_P h(x-y)\,dx\,dy$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \underbrace{I_{h_\ell}(P^{x_2}, P^{y_2})}_{\displaystyle \int_{P^{x_2}}\int_{P^{y_2}} h_\ell(x_1-y_1)\,dx_1\,dy_1}\,dx_2\,dy_2$$

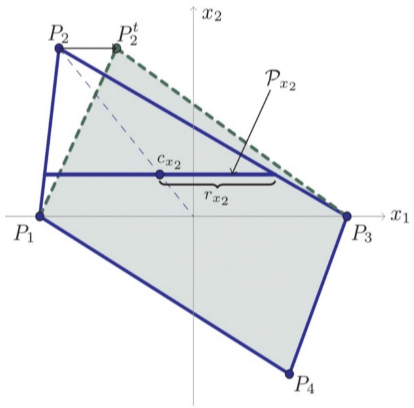
$$\left[\begin{array}{l} h_\ell(r) = h(\sqrt{r^2+\ell^2}) \\ \ell = |x_2-y_2| \end{array}\right]$$

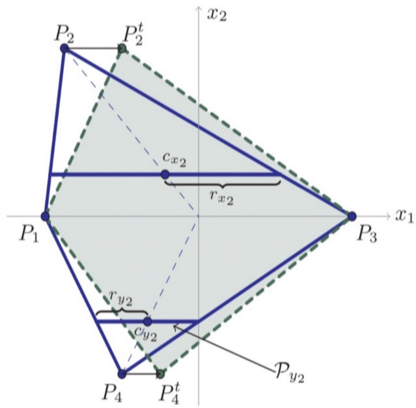
$$\Downarrow$$

$$\frac{d}{dt} J_h(P_t)\Big|_{t=0} = \int_{\mathbb{R}}\int_{\mathbb{R}} \underbrace{\frac{d}{dt} I_{h_\ell}(P_t^{x_2}, P_t^{y_2})}_{\displaystyle \geq c\min\{r_{x_2}, r_{y_2}\}\,|c_{x_2}-c_{y_2}|\,|x_2-y_2|}\,dx_2\,dy_2 \geq c' > 0$$

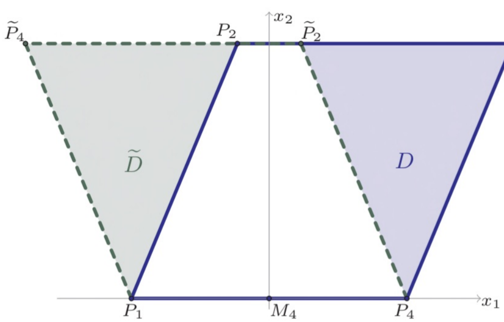
Proof (N=4)

By slices $\Rightarrow$ rhombus





By reflection $\Rightarrow$ square



∎

Open problems

① Prove rigidity for more general kernels, in particular not satisfying improved integrability

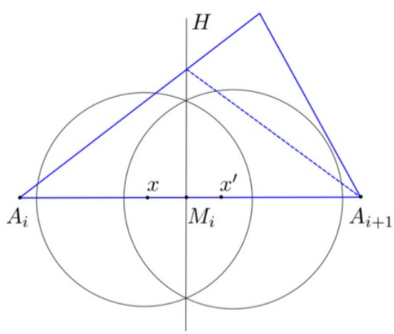
② Prove rigidity for polygons when $h = \chi_{B_r(0)}$:

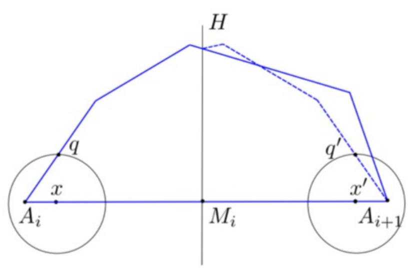
Given $r>0$, the regular N-gon is the unique N-gon such that

$$(*) \qquad \int_{A_i}^{A_{i+1}} |P \cap B_r(x)| \, d\mathcal{H}^1(x) = c\, \mathcal{H}^1(\overline{A_i A_{i+1}}) \qquad \forall i = 1, \dots, N$$

$$(**) \qquad \int_{A_i}^{M_i} |P \cap B_r(x)| \, |x - M_i| \, d\mathcal{H}^1 = \int_{M_i}^{A_{i+1}} |P \cap B_r(x)| \, |x - M_i| \, d\mathcal{H}^1 \qquad \forall i = 1, \dots, N$$

- True for $N=3$
- True for every N if r is small enough





(3) Prove some stability result for sets with "almost constant" nonlocal mean curvature.

Stability of Alexandrov type results

- [Ciraolo-Maggi CPAM 2017, Ciraolo-Vezzoni JEMS 2018]

  STABILITY OF CLASSICAL ALEXANDROV

- [Ciraolo-Figalli-Maggi-Novaga, J. Reine Angew. Math 2018]

  STABILITY IN THE FRACTIONAL CASE

Quantitative Riesz type inequalities

- [Christ, ArXiv Preprint 2017]
- [Frank-Lieb, ArXiv Preprint 2019 & Ann. SNS Pisa 2021]

Many Thanks

for your attention!

→ For any comment/question, e-mail to:

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