Polyhedral-like approximations in complex analysis

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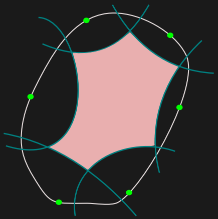

Online Asymptotic Geometric Analysis Seminar
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Notions of Complex Convexity

In Möbius geometry

$E \subset \mathbb{C}^d \subset \mathbb{CP}^d$, compact or open

$\mathbb{C}$-linear convexity

- $\mathbb{C}^d \setminus E$ is a union of complex hyperplanes.
- Preserved under intersections.
- Preserved under Cartesian products.
- Dual complement
$$E^* = \{\zeta \in \mathbb{C}^d : \langle z, \zeta \rangle \neq 1 \; \forall z \in E\}.$$
- $d = 1$: no condition.

$\mathbb{C}$-convexity

- $E \cap \ell$ is simply connected $\forall$ $\mathbb{C}$-lines ℓ.
- $\mathbb{C}$-convexity $\Rightarrow$ $\mathbb{C}$-linear convexity. Converse not true, even assuming connectedness.
- (Largely) not preserved under intersections & Cartesian products.
- E^* is $\mathbb{C}$-convex.
- open $E \cong$ ball.

⋆ Allow for Cauchy-type integral representations of holomorphic functions.

⋆ Invariant under automorphisms of $\mathbb{CP}^d$ (LFTs):
$$(z_1, \ldots, z_d) \mapsto \left(\frac{c_{11} z_1 + \cdots + c_{1d} z_d}{c_{01} z_1 + \cdots + c_{0d} z_d}, \ldots, \frac{c_{d1} z_1 + \cdots + c_{dd} z_d}{c_{01} z_1 + \cdots + c_{0d} z_d} \right).$$

⋆ If $E \subset \mathbb{C}^d$ is a $\mathcal{C}^1$-domain, $\mathbb{C}$-linear convexity $\iff$ $\mathbb{C}$-convexity.

In biholomorphic geometry

$E \subset \mathbb{C}^d$, open

Pseudoconvexity (holomorphic convexity)

- For every compact $K \subset E$, its *holomorphic hull*

$$\hat{K}_E = \left\{ z \in E : |f(z)| \leq \sup_K |f| \quad \forall f : \Omega \xrightarrow{\text{holo.}} \mathbb{C} \right\}$$

  is compact.
- Characterization of domains where simultaneous analytic extension doesn't occur.
- Non-example: $\mathbb{B}^d \setminus \frac{1}{2}\overline{\mathbb{B}}^d$.
- $d = 1$: all domains, as $\hat{K}_E$ "plugs" holes of K in E.
- Preserved under (open) intersections & products.
- Preserved by biholomorphisms.

⋆ Convexity $\Rightarrow$ $\mathbb{C}$-convexity $\Rightarrow$ $\mathbb{C}$-linear convexity $\Rightarrow$ psuedoconvexity

⋆ $E \subset \mathbb{R}^d = \{z \in \mathbb{C}^d : \text{Im}(z) = 0\}$

- E is convex $\iff$ E is $\mathbb{C}$-convex.
- E is convex $\iff$ $E + i\mathbb{R}^d$ is pseudoconvex.

Complex convexity for smooth domains

$\Omega \subset \mathbb{C}^d \cong \mathbb{R}^{2d}$, $\mathcal{C}^2$-smooth domain

$r : \mathbb{C}^d \to \mathbb{R}$: defining function

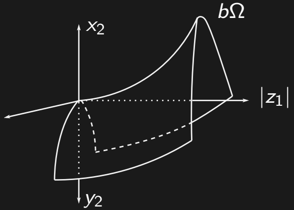
$p \in b\Omega$

T_p: real tangent space of $b\Omega$ at p

$H_p = T_p \cap iT_p$: complex tangent space of $b\Omega$ at p

II_p: real Hessian of r at p

L_p: complex Hessian of r at p



Convexity	$\mathbb{C}$-convexity	ψ-convexity
$\mathrm{II}_p\big\|_{T_p} \geq 0$	$\mathrm{II}_p\big\|_{H_p} \geq 0$	$L_p\big\|_{H_p} \geq 0$
$\Omega \cap T_p = \emptyset$	$\Omega \cap H_p = \emptyset$	No analogue

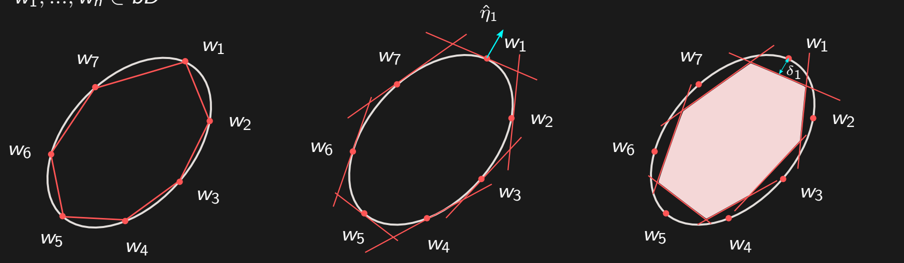
Strong convexity	Strong $\mathbb{C}$-convexity	Strong ψ-convexity
$\overline{\Omega} \cap T_p = \{p\}$	$\overline{\Omega} \cap H_p = \{p\}$	Local quadratic analogue
affine $\cong$ ball	LFT $\cong$ ball	bihol. $\cong$ ball
$y_2 > x_1^2 + y_1^2 + x_2^2$	$y_2 > x_1^2 + y_1^2 - x_2^2$	$y_2 > 2x_1^2 - y_1^2 - x_2^2$

(Best) Polyhedral Approximations

In $\mathbb{R}^d$: schemes of approximation

D: convex domain

$w_1, \ldots, w_n \in bD$



$P = \mathrm{conv}\{w_1, \ldots, w_n\}$

$\mathcal{P}^{\mathrm{i}}_n(D) = \{$inscribed poly. with $\leq n$ vertices$\}$

$P = \bigcap \{\langle \hat{\eta}_{w_j}, z - w_j \rangle \leq 0\}$

$\mathcal{P}^{\mathrm{c}}_{(n)}(D) = \{$circumscribed poly. with $\leq n$ facets$\}$

$P = \bigcap \{\langle \hat{\eta}_{w_j}, z - w_j \rangle \leq -\delta_j\}$

$\mathcal{P}^{\mathrm{co}}_{(n)}(D) = \{$contained poly. with $\leq n$ facets$\}$

Efficacy of the approximation:

- $\delta_{\mathrm{V}}(D, P) = \mathrm{vol}(D \triangle P)$
- $\delta_{\mathrm{H}}(D, P) =$ Hausdorff distance between D & P

In $\mathbb{R}^d$: typical results

- Optimal approximation asymptotics.

$$\inf\{\delta(D,P) : \text{complexity}(P) \leq n\} \sim C_{d,D}\frac{1}{n^{k(d)}} \quad \text{as } n \to \infty.$$

- Identifying "almost-optimal" polyhedra.

  Distribution of the source points w_j of "best" polyhedra:

  - uniform with respect to certain densities,
  - centers of minimal ball coverings of bD in some metric

  Asymptotic shapes of the facets

- Random approximation asymptotics.

Given i.i.d. random source points $W_1, ..., W_n \sim h$ on bD,

$$\delta(D;P) \overset{p,\ L^1,\ a.s.}{\sim} C_{d,D,h}\frac{\log(n)^{\ell(d)}}{n^{k(d)}} \quad \text{as } n \to \infty.$$

In $\mathbb{R}^d$: some optimal approximation results

Gruber (1993), Ludwig (1999). Let $D \in \mathbb{R}^d$ be a strongly convex $\mathcal{C}^2$ domain.

$$v_n^{\mathrm{c}} := \inf\left\{\mathrm{vol}(P \setminus D) : P \in \mathcal{P}_{(n)}^{\mathrm{c}}(D)\right\} \sim a_d \cdot \sigma_B(bD)^{\frac{d+1}{d-1}} \cdot \frac{1}{n^{2/(d-1)}} \qquad \text{as } n \to \infty.$$

$$v_n := \inf\left\{\mathrm{vol}(D \triangle P) : P \in \mathcal{P}_{(n)}(D)\right\} \sim b_d \cdot \sigma_B(bD)^{\frac{d+1}{d-1}} \cdot \frac{1}{n^{2/(d-1)}} \qquad \text{as } n \to \infty.$$

- $a_d = \mathrm{div}_{d-1}$ and $b_d = \mathrm{ldiv}_{d-1}$ are unknown for $d > 2$.
- The Blaschke measure on bD: $\sigma_B = \kappa^{1/(d+1)}\sigma_E$, where

$$\begin{aligned} \kappa &= \text{Gaussian curvature function on } bD, \\ \sigma_E &= \text{Euclidean surface area measure on } bD. \end{aligned}$$

- Among bodies of unit volume, ellipsoids are the "hardest" to approximate!
- Böröczky (2000) removed the curvature assumption.

Geometric & combinatorial aspects of the problem

Transformation Geometry

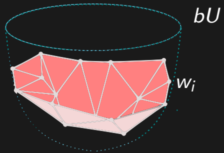
- (Strong) Convexity, classes of polyhedra: invariant under affine tranformations of $\mathbb{R}^d$.
- v_n^c and v_n: invariant under volume-preserving or equi-affine transformations of $\mathbb{R}^d$.
- Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be an affine map, and $D' = A(D)$. Then

$$A^* \sigma_B^{D'} = |\det A|^{\frac{d-1}{d+1}} \sigma_B^{D}.$$

Tilings on $\mathbb{R}^{d-1}$

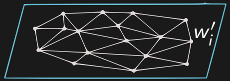
2nd-order local model for strongly convex domains:

$$U = \left\{ (x_1, \dots, x_d) \in \mathbb{R}^d : x_d > x_1^2 + \cdots x_{d-1}^2 \right\}.$$



$$\mathbf{w} = \{w_1, \dots, w_n\} \in bU \xrightarrow{proj.} \mathbf{w}' = \{w_1', \dots, w_n'\} \in \mathbb{R}^{d-1}$$

- div_{d-1}: facets of $\mathrm{circ}\{\mathbf{w}\} \xrightarrow{proj.}$ Dirichlet–Voronoi cells of $\mathbf{w}'$.
- ldiv_{d-1}: facets of $P(\mathbf{w}, \boldsymbol{\delta}) \xrightarrow{proj.}$ Laguerre cells of $(\mathbf{w}', \boldsymbol{\delta})$.



Dual Image

Polyhedral constructions in $\mathbb{C}^d$

- No notion of $\mathbb{C}$-convex hulls or psuedoconvex hulls for finite sets!

- In the literature: an *analytic polyhedron in Ω with $\leq n$ facets* is any finite union of relatively compact components of

$$\left\{z \in \Omega : |f_j(z)| < 1,\ j = 1, ..., n\right\}, \quad f_j : \Omega \xrightarrow{\text{hol.}} \mathbb{C}.$$

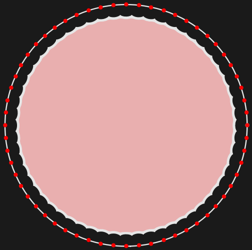
Bishop (1961). Any bounded ψ-convex domain in $\mathbb{C}^d$ can be approximated arbitrarily well by d-faceted analytic polyhedra.

$\Omega = \mathbb{D}$

$$P_m := \left\{ z \in \mathbb{D} : \prod_{k=0}^{2m-1} \left| z - \exp\left(\frac{k\pi i}{m}\right)\right| > \frac{\pi}{m} \right\}$$

- $\inf\{\text{vol}(\mathbb{D} \setminus P) : P \text{ has one facet}\} = 0.$
- $m\ \text{vol}(\mathbb{D} \setminus P_m) \to c \neq 0$ as $m \to \infty$.

Want to say: P_m has $\mathrm{O}(m)$ facets.



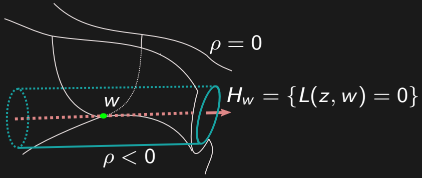
$P_{30} \subset \mathbb{D}$

- We will mimic the "pushing in" of tangent planes.

Polyhedral constructions in $\mathbb{C}^d$

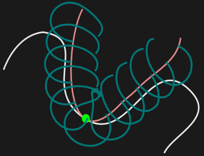
A convex polyhedron in $D \subset \mathbb{R}^d$: $\bigcap_{1\le j\le n} \{\overbrace{\left\langle \hat{\eta}_{w^j}, z - w^j \right\rangle}^{f(z,w^j)} < \delta_j\}$

$\Omega = \{\rho < 0\}$ is strongly $\mathbb{C}$-convex.

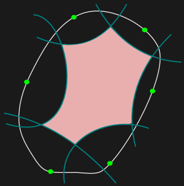


- $L(z,w) = \sum \frac{\partial \rho}{\partial z_j}(w)(z_j - w_j)$
- $H^+(w,\delta) = \{z \in \Omega : |L(z,w)| > \delta\}$

$\Omega = \{\rho < 0\}$ is strongly ψ-convex.



- $Q_\rho(z,w) = L(z,w) +$ 2nd order terms
- $H^+(w,\delta) = \{z \in \Omega : |Q_\rho(z,w)| > \delta\}$



$\boldsymbol{w} = \{w^1, \dots, w^n\} \subset b\Omega$ (source set)

$\boldsymbol{\delta} = \{\delta_1, \dots, \delta_n\} \subset \mathbb{R}_+$ (depth set)

$$P(\boldsymbol{w};\boldsymbol{\delta}) := \bigcap_{1\le j\le n} H^+(w^j, \delta_j)$$

$$\mathcal{P}_n(\Omega) = \{P(\boldsymbol{w};\boldsymbol{\delta}) : P(\boldsymbol{w};\boldsymbol{\delta}) \Subset \Omega\}$$

Some relevant features of complex convexity ($d > 1$)

$\sum = \sum_{j=1}^{d-1}$

$\Omega \Subset^{\mathcal{C}^2 \text{domain}} \mathbb{C}^d$	Strong $\mathbb{C}$-convexity	Strong pseudoconvexity
polyhedra	Leray polyhedra	Levi polyhedra
Transform. grp.	LFTs/Möbius	biholomorphisms
Local model(s)	$\operatorname{Im} z_d > \sum \lvert z_j\rvert^2 + \sum \beta_j \operatorname{Re}(z_j)^2$	$\operatorname{Im} z_d > \lvert z_1\rvert^2 + \cdots + \lvert z_{d-1}\rvert^2$
2nd order inv.	Eccentricity $\boldsymbol{\beta}(p) = (\beta_1, \ldots, \beta_{d-1})$	None
"Equi"-models	$\operatorname{Im} z_d > \sum \alpha_j \lvert z_j\rvert^2 + \sum \gamma_j \operatorname{Re}(z_j)^2$	$\operatorname{Im} z_d > \alpha_1 \lvert z_1\rvert^2 + \cdots + \alpha_{d-1}\lvert z_{d-1}\rvert^2$
"Equi"-inv.	$\alpha_j > \gamma_j \geq 0$	$\alpha_j > 0$
Gaussian analogue	$\mathcal{Q}(p) = \prod(\alpha_j^2 - \gamma_j^2)$ $= \mathcal{L}^2(p) \prod(1 - \beta_j^2)$	Levi curvature $\mathcal{L}(p) = \alpha_1 \cdots \alpha_{d-1}$
Blaschke analogue $\sigma_B = \kappa^{1/d} \sigma_E$	Möbius–Fefferman $\sigma_{MF} = \mathcal{Q}^{1/(2d+2)} \sigma_E = G(\boldsymbol{\beta}) \sigma_F$	Fefferman $\sigma_F = \mathcal{L}^{1/(d+1)} \sigma_E$

Optimal approximation results

Theorem (G., 2017, 2023+). Let $\Omega \Subset \mathbb{C}^d$ be a $\mathcal{C}^\infty$-smooth domain.

1. Ω is strongly pseudoconvex. $\exists\ k_d > 0$ s.t.

$$\inf\{\mathrm{vol}(D \setminus P) : P \in \mathcal{P}_n(\Omega)\} \sim k_d \cdot \sigma_F(b\Omega)^{\frac{d+1}{d}} \cdot \frac{1}{n^{1/d}} \qquad \text{as } n \to \infty.$$

2. Ω is strongly $\mathbb{C}$-convex. $\exists$ continuous $K_d : [0,1)^{d-1} \to (0,\infty)$ s.t.

$$\inf\{\mathrm{vol}(D \setminus P) : P \in \mathcal{P}_n(\Omega)\} \sim k_d \cdot \left(\int_{b\Omega} K_d(\beta(z))\, d\sigma_F(z)\right)^{\frac{d+1}{d}} \cdot \frac{1}{n^{1/d}} \qquad \text{as } n \to \infty.$$

$d = 1$. Each Leray "cut" is a disk.

$$v_n \sim \frac{\pi}{8} \cdot (\sigma_{\mathrm{arc}}(b\Omega))^2 \cdot \frac{1}{n} \qquad \text{as } n \to \infty.$$

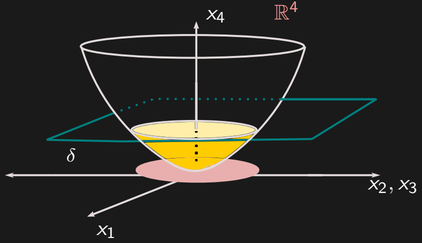
$\beta \equiv 0$. $\Omega \overset{LFT}{\cong} \mathbb{B}^d$ and $K_d(0, \ldots, 0) = 1$.

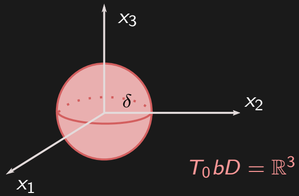
Speculation. The measure $K_d(\beta)\,\sigma_F$ is σ_{MF}, i.e., $K_d(\beta)^{d+1} = \sqrt{\prod(1-\beta_j^2)}$.

A single “cap”

Model in $\mathbb{R}^4$: $\{x_4 > x_1^2 + x_2^2 + x_3^2\}$

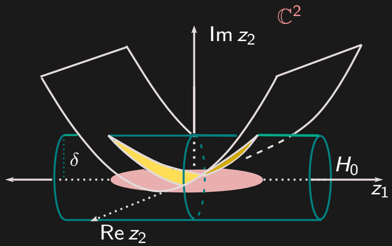
Projection of δ-cap at $(0,0) = \sqrt{\delta}$-ball in Euclidean metric on $(\mathbb{R}^3, +)$.

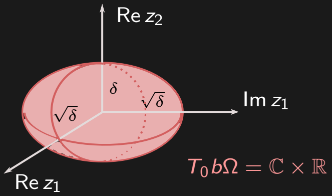




Model in $\mathbb{C}^2$: $\{\mathrm{Im}\, z_2 > |z_1|^2\}$

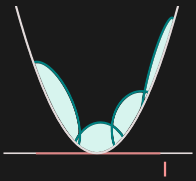
Projection of δ-cap at $(0,0) = \sqrt{\delta}$-ball in Korányi metric on $(\mathbb{C} \times \mathbb{R}, \text{Heisenberg})$.





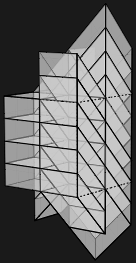
The models $\leftrightarrow$ 'good' tilings of the Heisenberg group ($d = 2$)

$$D_\beta = \left\{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Im} z_2 > |z_1|^2 + \beta \operatorname{Re} z_1^2\right\}, \quad \beta \in [0, 1)$$



- I = unit cube in $\mathbb{C} \times \mathbb{R}$
- $C_\beta(w, \delta)$ = Leray cut with source w and depth δ
- $c_\beta(w, \delta)$ = projection of $C(w, \delta)$
- $v_n = \inf \left\{ \operatorname{vol} \bigcup_{j=1}^{n} C(w^j, \delta_j) : \mathrm{I} \subset \bigcup_{j=1}^{n} c(w^j, \delta_j) \right\}$

Claim. $\lim_{n\to\infty} \sqrt{n} v_n$ exists $=: 2k_2 \cdot K_2(\beta)^{3/2}$.



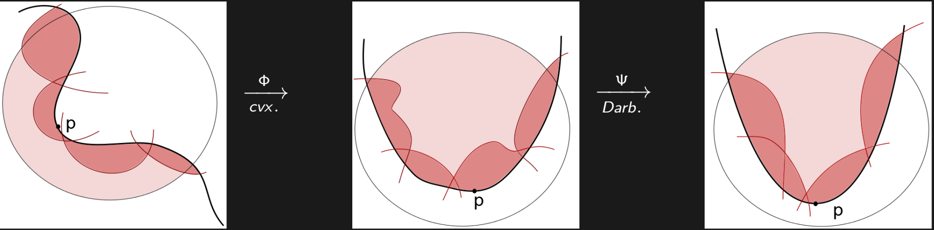
- Key ingredient: $\exists$ on $\mathbb{C} \times \mathbb{R}$
  * group operation $\otimes_\beta$: $w^2 \otimes_\beta c(w^1; \delta) = c(w^2 \otimes_\beta w^1; \delta)$.
  * left-invariant quasimetric d_β: $c(w; \delta) = \left\{ d_\beta(w, z) < \sqrt{\delta} \right\}$.
- $K_2(\beta)$ comes from exploiting d_β-tilings of $\mathbb{C} \times \mathbb{R}$.

Missing. $K_2(\beta) = (1 - \beta)^{3/2}$.

- All $(\mathbb{C} \times \mathbb{R}, \otimes_\beta)$ are isomorphic (to the Heisenberg group).
- These isomorphisms are not isometries!

From the model to the general case

The technique of ‘shaking’ is entirely unavailable!



Near p, Φ and Ψ must

- be close to volume-preserving;
- be close to s_{Euc}-preserving on $\partial\Omega$;
- keep the pushed-forward cuts and model cuts ‘comparable’.

The maps:

- Φ is an almost explicit LFT.
- The boundary of a strongly $\mathbb{C}$-convex Ω has a natural contact structure.
- Darboux: any two equi-dim. contact str. are loc. contact isomorphic.
- Ψ along $\partial\Omega$ is a Darboux map.

On the exponents

1. $D \subset \mathbb{R}^d$.

$$v_n \sim c_d \cdot (measure)^{\frac{d+1}{d-1}} \cdot \frac{1}{n^{2/(d-1)}} \qquad \text{as } n \to \infty.$$

2. $\Omega \subset \mathbb{C}^d$.

$$v_n \sim b_d \cdot (measure)^{\frac{d+1}{d}} \cdot \frac{1}{n^{1/d}} \qquad \text{as } n \to \infty.$$

Exponents	Haus. dim. of induced metric	measure	complexity
in $\mathbb{R}^d$	$d-1$	$(d+1)/(d-1)$	$2/(d-1)$
in $\mathbb{C}^d$	$2d$	$(d+1)/d$	$1/d$
	$=\eta$	$=(\eta+2)/\eta$	$=2/\eta$

(Random) Polyhedral Approximations

In $\mathbb{R}^d$: some random approximation results

$D \Subset \mathbb{R}^d$ $\mathcal{C}^2$, strongly convex (!)

$X^1, ..., X^n \in bD$ are i.i.d. with density $f : bD \xrightarrow{(!)} \mathbb{R}_+$

Schütt–Werner (2003). $P_n = \mathrm{conv}\left\{X^1, ..., X^n\right\}$.

$$n^{\frac{2}{d-1}}\, \mathbb{E}(\delta_V(D, P_n)) \to c_1(d, D, f) \qquad \text{as } n \to \infty.$$

Böröczky–Reitzner (2004). $P_n = \bigcap_{j=1}^{n} H^+(X^i) \bigcap$ large ball.

$$n^{\frac{2}{d-1}}\, \mathbb{E}(\delta_V(D, P_n)) \to c_2(d, D, f) \qquad \text{as } n \to \infty.$$

Glasauer–Schneider (1996). $P_n = \mathrm{conv}\{X_1, ..., X_n\}$.

$$\left(\frac{n}{\log(n)}\right)^{\frac{2}{d-1}} \delta_H(D, P_n) \xrightarrow{p} c_3(d, D, f) \qquad \text{as } n \to \infty.$$

- Exponent of $n =$ that in the optimal case.
- The best density $=$ (normalized) boundary measure in the optimal case.
- Best "random" constant differs "optimal" constant only by a dimensional factor.

Random polyhedra in strongly $\mathbb{C}$-convex domains ($d > 1$)

(Joint work with S. Athreya & D. Yogeshwaran)

Domain. $\Omega \Subset \mathbb{C}^d$: strongly $\mathbb{C}$-convex $\mathcal{C}^2$ domain.

Random (Leray) polyhedron. $P_n := P(\boldsymbol{w}_n; \boldsymbol{\delta}_n)$, where

- $\boldsymbol{w}_n = \{W^1, ..., W^n\} \subset b\Omega$, $W^1, ..., W^n$ are i.i.d. with density $f : b\Omega \xrightarrow{cont.} (0, \infty)$.
- $\boldsymbol{\delta}_n : b\Omega \xrightarrow{cont.} \mathbb{R}_+$ with appropriate decay.

$$P_n = \bigcap_{1 \leq j \leq n} H^+ \left(W^j, \boldsymbol{\delta}_n(W^j) \right).$$

Metric of approximation. $\delta_V(n) := \text{vol}(\Omega \setminus P_n)\mathbf{1}(P_n \Subset \Omega) + \text{vol}(\Omega)\mathbf{1}(P_n \not\Subset \Omega)$

$*$ In $\mathbb{R}^d$, such a "penalty" is imposed when circumscribing by random polyhedra.

The depth function.

$$\boldsymbol{\delta}_n(z) = \left(\frac{\log(n)}{n} \right)^{\frac{1}{d}} g(z), \quad z \in b\Omega,$$

for $g : b\Omega \xrightarrow{cont.} \mathbb{R}_+$ such that

$$\lim_{n \to \infty} \mathbb{P}(P_n \Subset \Omega) = 1. \qquad (\star)$$

$*$ Log factor: $P_n \Subset \Omega \iff$ the "caps" of P_n cover $b\Omega$.

$*$ The decay rate and $(\star)$ are compatible.

A random approximation result

Theorem (Athreya-G.-Yogeshwaran, 2022). Given Ω, f, g, $\boldsymbol{w}_n$ and $\boldsymbol{\delta}_n$ as above

$$\left(\frac{n}{\log(n)}\right)^{\frac{1}{d}} \delta_V(n) \xrightarrow{p} \int_{b\Omega} g(z) d\sigma_{Euc}(z) \qquad \text{as } n \to \infty.$$

Optimal random approximation?

Q1. What is the best R.H.S., say $\nu_D(f)$, for a fixed f?

Missing. The Leray polyhedra are associated to a natural sub-Riemannian metric d on $b\Omega$. We need asymptotics of

$$R_n = \min\left\{ r > 0 : b\Omega \subset \bigcup_{j=1}^{n} B_d(W^j, r) \right\}.$$

Q2. Which density f gives $\nu_D :=$ least possible $\nu_D(f)$?

Conjecture. Assuming heuristics for R_n,

$$f\sigma_{Euc} = \frac{\sigma_{MF}}{\sigma_{MF}(b\Omega)} \qquad \text{and} \qquad \nu_D = \widetilde{k}_d \left(\sigma_{MF}(b\Omega)\right)^{\frac{d+1}{d}}.$$

THANK YOU.