An intrinsic volume metric for convex bodies

Online Asymptotic Geometric Analysis Seminar

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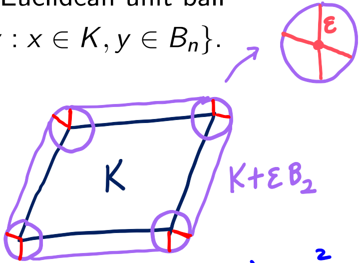
Background

The **intrinsic volumes** $V_0(K), V_1(K), \ldots, V_n(K)$ of a convex body K in $\mathbb{R}^n$ are defined as the coefficients in *Steiner's formula* for the volume of the outer parallel body

$$\mathrm{vol}_n(K + \varepsilon B_n) = \sum_{j=0}^{n} \varepsilon^{n-j} \,\mathrm{vol}_{n-j}(B_{n-j}) V_j(K) \qquad \forall \varepsilon \geq 0,$$

where B_m denotes the m-dimensional Euclidean unit ball centered at o and $K + \varepsilon B_n = \{x + \varepsilon y : x \in K, y \in B_n\}$.

- $V_n(K) = \mathrm{vol}_n(K)$
- $V_{n-1}(K) = \frac{1}{2}\,\mathrm{vol}_{n-1}(\partial K)$
- $V_1(K) = c(n) w(K)$
- $V_0(K) = \chi(K) = 1$

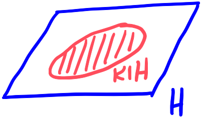


$\mathrm{area}(K+\varepsilon B_2) = \mathrm{area}(K) + \varepsilon \cdot \mathrm{per}(K) + \pi \varepsilon^2$
$= \varepsilon^2 \pi V_0(K) + 2\varepsilon V_1(K) + V_2(K)$

Kubota's integral formula

For $j \in [n] = \{1, \ldots, n\}$,

$$V_j(K) = \begin{bmatrix} n \\ j \end{bmatrix} \int_{\mathrm{Gr}(n,j)} \mathrm{vol}_j(K|H)\,\mathrm{d}\nu_j(H)$$



where:

- $\mathrm{Gr}(n,j)$ is the Grassmannian of all j-dimensional subspaces of $\mathbb{R}^n$, and ν_j is the (uniquely determined) Haar probability measure on $\mathrm{Gr}(n,j)$;
- $K|H$ is the orthogonal projection of K into the subspace $H \in \mathrm{Gr}(n,j)$;
- $\begin{bmatrix} n \\ j \end{bmatrix} = \binom{n}{j} \frac{\mathrm{vol}_n(B_n)}{\mathrm{vol}_j(B_j)\,\mathrm{vol}_{n-j}(B_{n-j})} = \frac{1}{2} \frac{\omega_{j+1}\omega_{n-j+1}}{\omega_{n+1}}$ is the *flag coefficient* of Klain and Rota (1997), where $\omega_n = n\,\mathrm{vol}_n(B_n)$.

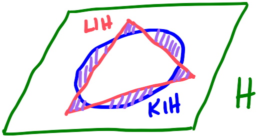
The intrinsic volume metric

Definition

For convex bodies K and L in $\mathbb{R}^n$ and $j \in [n]$, we define the jth **intrinsic volume metric** δ_j by

$$\delta_j(K, L) := \begin{bmatrix} n \\ j \end{bmatrix} \int_{\mathrm{Gr}(n,j)} \mathrm{vol}_j((K|H) \triangle (L|H)) \, \mathrm{d}\nu_j(H).$$

- This quantity may be thought of as the mean distance of the shadows of K and L, averaged over all j-dimensional subspaces.
- Note that $\delta_n(K, L) = \mathrm{vol}_n(K \triangle L)$.



Properties of δ_j

Theorem (Besau-H., 2023)

The functional $\delta_j : \mathcal{K}^n \times \mathcal{K}^n \to [0, \infty)$ *is:*

(i) *a metric on* $\mathcal{K}^n$*;*

(ii) *continuous with respect to the Hausdorff metric;*

(iii) *rigid motion invariant, that is,*

$$\delta_j(\vartheta K + x, \vartheta L + x) = \delta_j(K, L)$$

for all orthogonal transformations $\vartheta \in O(\mathbb{R}^n)$ *and all* $x \in \mathbb{R}^n$*;*

(iv) *positively* j*-homogeneous, that is,*

$$\delta_j(tK, tL) = t^j \delta_j(K, L), \quad \forall t > 0.$$

Comparison with other intrinsic volume distances

- Florian (1989): $\rho_j(K,L) = 2V_j([K,L]) - V_j(K) - V_j(L)$
- Besau-H.-Kur (2019):
  $\Delta_j(K,L) = V_j(K) + V_j(L) - 2V_j(K \cap L)$

Proposition (Besau-H., 2023)

For all convex bodies $K, L \in \mathcal{K}^n$, we have:

(i) *If $K \subset L$ and $j \in [n]$, then*
$\delta_j(K,L) = \rho_j(K,L) = \Delta_j(K,L) = V_j(L) - V_j(K)$*;*

(ii) $\delta_j(K,L) \le \min\{\rho_j(K,L), \Delta_j(K,L)\}$ *for all $j \in [n]$;*

(iii) $\delta_j(K,L) \ge |V_j(K) - V_j(L)|$ *for all $j \in [n]$;*

(iv) $\rho_n(K,L) \ge \delta_n(K,L) = \Delta_n(K,L) = \mathrm{vol}_n(K \triangle L)$*;*

(v) *If $K \cap L \neq \varnothing$, then*

$$\begin{aligned}\Delta_1(K,L) \ge \delta_1(K,L) &= \rho_1(K,L) \\ &= 2\left[{n \atop 1}\right] \int_{\mathbb{S}^{n-1}} |h_K(u) - h_L(u)| \,\mathrm{d}\sigma(u).\end{aligned}$$

The random beta polytope model

- Let $X_1, \ldots, X_N$ be i.i.d. points chosen from $\mathbb{R}^n$ according to the beta distribution, which for a parameter $\beta > -1$ has the density

$$f_{n,\beta}(x) = \frac{\Gamma\left(\frac{n}{2} + \beta + 1\right)}{\pi^{\frac{n}{2}}\Gamma(\beta + 1)}(1 - \|x\|^2)^{\beta}\mathbb{1}_{\{x:\|x\|<1\}}(x).$$

- A **random beta polytope** $P_{n,N}^{\beta}$ is the convex hull of the X_i, which is denoted by $[X_1, \ldots, X_N]$.
- The uniform probability distribution on B_n is the $\beta = 0$ distribution.
- Kabluchko, Temesvari and Thäle (2019): The uniform probability distribution σ on the sphere $\mathbb{S}^{n-1}$ is the weak limit of the beta distribution as $\beta \to -1^+$.

Expected volume of random beta polytopes

Theorem (Affentranger, 1991)

Let $n \in \mathbb{N}$ and let $X_1, \ldots, X_N$ be i.i.d. random points chosen from B_n according to the beta distribution with $\beta > -1$, and set $P_{n,N}^{\beta} := [X_1, \ldots, X_N]$. Then the expected volume of $P_{n,N}^{\beta}$ satisfies

$$\lim_{N \to \infty} N^{\frac{2}{n+2\beta+1}} \mathbb{E}[\mathrm{vol}_n(B_n \setminus P_{n,N}^{\beta})] = A_{n,\beta},$$

where

$$\begin{aligned} A_{n,\beta} &:= \frac{\omega_n}{2} \frac{n+2\beta+1}{n+2\beta+3} \frac{\Gamma\left(n+1+\frac{2}{n+2\beta+1}\right)}{\Gamma(n+1)} d_{n,\beta}^{\frac{2}{n+2\beta+1}} \\ &= \frac{\omega_n}{2} \left(1 + O\left(\frac{\ln(n+2\beta+2)}{n+2\beta+1}\right)\right), \quad \forall n \in \mathbb{N},\ \forall \beta \in [-1/2, \infty) \end{aligned}$$

and $d_{n,\beta}$ is a specifically known constant depending on the second moment of a random beta simplex inscribed in B_{n-1}.

Asymptotic best approximation of the ball

Theorem (Besau-H.-Kur, 2019)

For every $j \in [n]$ there exist absolute constants $c_1, c_2 > 0$ such that for all sufficiently large N, there exists a polytope $P_{n,j,N} \subset B_n$ with at most N vertices (respectively, $P_{n,j,N} \supset B_n$ with at most N facets) which satisfies

$$c_1 j V_j(B_n) N^{-\frac{2}{n-1}} \leq \Delta_j(B_n, P_{n,j,N}) \leq c_2 j V_j(B_n) N^{-\frac{2}{n-1}}.$$

- It is also shown that $c_1 \sim c_2 = \frac{1}{2} + O\left(\frac{\ln n}{n}\right)$ as $n \to \infty$.
- Interestingly, it turns out there is a polytope which satisfies all n inequalities (5) simultaneously.

Comparing best and random approximations of the ball

Comparing the previous two theorems, we find that in the inscribed case, random approximation of the ball is asymptotically (almost) as good as best approximation.

Corollary (Besau-H.-Kur, 2019)

Choose N points $X_1, \ldots, X_N$ independently with respect to the uniform probability measure σ on the unit sphere $\mathbb{S}^{n-1}$, and let $P_N := [X_1, \ldots, X_N]$. Then for every $j \in [n]$,

$$\limsup_{N\to\infty} \frac{\mathbb{E}[\Delta_j(B_n, P_N)]}{\Delta_j(B_n, P_N^{\text{best}})} = 1 + O\left(\frac{\ln n}{n}\right).$$

Arbitrarily positioned polytopes: Volume, surface area and mean width approximations

- For the symmetric difference metric ($j = n$) and surface area deviation ($j = n - 1$), dropping the restriction that the ball contains the polytope (or vice versa) improves the estimate by at least a factor of n.
- The same phenomenon has also been observed for the mean width metric ρ_1.
- This can be seen by comparing results of Besau-H.-Kur (2019), Glasauer-Gruber (1997), Grote-Thäle-Werner (2021), Grote-Werner (2018), Gruber (1993), H.-Kur (2021), H.-Schütt-Werner (2018), Ludwig (1999), Ludwig-Schütt-Werner (2006) and Kur (2020).

Arbitrarily positioned polytopes: Intrinsic volume approximation

Theorem (Besau-H.-Kur, 2019)

There exists an absolute constant C such that for all sufficiently large N,

$$\min_{Q\in\mathcal{P}_{n,N}} \Delta_j(B_n, Q) \le C \min\left\{1, \frac{j \ln n}{n}\right\} V_j(B_n) N^{-\frac{2}{n-1}}. \qquad (1)$$

where $\mathcal{P}_{n,N}$ is the set of all polytopes in $\mathbb{R}^n$ with at most N vertices.

- Recall that $\delta_j \le \Delta_j$.

  How much can we improve the upper bound (1) for the approximation if we measure the distance by δ_j instead of Δ_j?

Main result

Theorem (Besau-H., 2023)

There exists an absolute constant C such that for every $n \in \mathbb{N}$ with $n \geq 2$ and every $j \in [n]$, when N is sufficiently large

$$\min_{Q \in \mathcal{P}_{n,N}} \delta_j(B_n, Q) \leq C \frac{j}{n} V_j(B_n) N^{-\frac{2}{n-1}}. \tag{2}$$

It is shown that $C = 2 + O\left(\frac{\ln n}{n}\right)$ as $n \to \infty$.

Main ingredients of the proof

The proof combines ideas from two papers:

- The random construction of Ludwig-Schütt-Werner (2006) yields the best-known estimate for the asymptotic best approximation of B_n by polytopes with N vertices in the symmetric difference metric.
- A random uniform polytope is generated in the ball, which is shrunk by a carefully chosen factor depending on N. The expected symmetric volume difference is estimated using the Blaschke-Petkanschin formula.
- The orthogonal projection of a beta distribution onto a subspace yields another beta distribution. Kabluchko-Temesvari-Thäle (2019) give a formula for it.

Step 1: Reduction to the weighted symmetric volume difference

First we reduce the problem to estimating the expected symmetric volume difference of the projection of a random beta polytope and a Euclidean ball.

Lemma (Besau-H., 2023)

Let $U_1, \ldots, U_N$ be chosen independently and uniformly from the sphere $\mathbb{S}^{n-1}$, and set $P_{n,N}^{\mathrm{unif}} := [U_1, \ldots, U_N]$. Then for any fixed $r > 0$ and all $j \in [n]$,

$$\mathbb{E}[\delta_j(P_{n,N}^{\mathrm{unif}}, rB_n)] = \begin{bmatrix} n \\ j \end{bmatrix} \mathbb{E}[\mathrm{vol}_j(P_{j,N}^{\beta=\frac{n-j-2}{2}} \triangle rB_j)].$$

Step 2: The choice of scaling factor

For any $r \in (0,1)$,

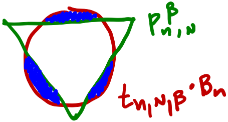
$$\mathbb{E}[\mathrm{vol}_n(rB_n \triangle P_{n,N}^{\beta})] = \mathrm{vol}_n(B_n \setminus rB_n) - \mathbb{E}[\mathrm{vol}_n(B_n \setminus P_{n,N}^{\beta})] + 2\mathbb{E}[\mathrm{vol}_n(rB_n \cap (P_{n,N}^{\beta})^c)]. \quad (3)$$

Given $N \geq n+1$ and $\beta \geq -1$, $\exists \gamma_{n,N,\beta} \in (0,1)$ such that

$$\mathrm{vol}_n(B_n \setminus (1-\gamma_{n,N,\beta})B_n) = \mathbb{E}[\mathrm{vol}_n(B_n \setminus P_{n,N}^{\beta})]. \quad (4)$$

Setting $r = t_{n,N,\beta} := 1 - \gamma_{n,N,\beta}$ and $\mathrm{d}\mathbb{P}_\beta(x) = f_{n,\beta}(x)\,\mathrm{d}x$, by (3) and (4) we have

$$\mathbb{E}[\mathrm{vol}_n(t_{n,N,\beta}B_n \triangle P_{n,N}^{\beta})] = 2\int_{B_n}\cdots\int_{B_n} \mathrm{vol}_n(t_{n,N,\beta}B_n \setminus [x_1,\ldots,x_N]) \times \mathrm{d}\mathbb{P}_\beta(x_1)\cdots\mathrm{d}\mathbb{P}_\beta(x_N).$$



Estimating the inflation factor

- By the choice of $\gamma_{n,N,\beta}$, Affentranger's result and the homogeneity of volume,

$$\gamma_{n,N,\beta} \sim \frac{\mathbb{E}[\mathrm{vol}_n(B_n \setminus P^{\beta}_{n,N})]}{n\,\mathrm{vol}_n(B_n)} \sim \frac{A_{n,\beta}}{\omega_n} N^{-\frac{2}{n+2\beta+1}}$$

as $N \to \infty$.

- By Stirling's inequality, $\exists c_1, c_2 > 0$ (absolute constants) such that

$$c_1 N^{-\frac{2}{n+2\beta+1}} \leq \gamma_{n,N,\beta} \leq c_2 N^{-\frac{2}{n+2\beta+1}}.$$

In fact,

$$\frac{A_{n,\beta}}{\omega_n} \sim c_1 \sim c_2 = \frac{1}{2}\left(1 + O\left(\frac{\ln(n+2\beta+2)}{n+2\beta+1}\right)\right)$$

as $n \to \infty$.

Step 3: The local estimate

The next result extends LSW from the uniform distribution on the sphere $\mathbb{S}^{n-1}$ $(\beta = -1)$ to all beta distributions on B_n with $\beta \geq -\frac{1}{2}$.

Theorem (Besau-H., 2023)

Fix $n \in \mathbb{N}$ and $\beta \geq -\frac{1}{2}$, and let $P_{n,N}^{\beta}$ be the convex hull of $N \geq n+1$ random points $X_1, \ldots, X_N$ chosen i.i.d. from the Euclidean unit ball B_n with respect to the beta distribution. Then for all sufficiently large N,

$$\mathbb{E}[\mathrm{vol}_n(B_n \triangle t_{n,N,\beta}^{-1} P_{n,N}^{\beta})] \leq \left(1 + O\left(\frac{\ln(n+2\beta+2)}{n+2\beta+1}\right)\right) \times \frac{2n\,\mathrm{vol}_n(B_n)}{n+2\beta+1} N^{-\frac{2}{n+2\beta+1}}.$$

Step 3(i): Reduction to beta polytopes containing o

- Choose i.i.d. random points $X_1, X_2, \ldots$ from B_n according to $f_{n,\beta}$, and for $N \geq n+1$ define $P_{n,N}^{\beta} := [X_1, \ldots, X_N]$.
- Let $\mathcal{E}_{n,N,\beta}$ denote the event that the origin o lies in the interior of $P_{n,N}^{\beta}$. By a result of Schütt and Werner (2003),

$$\begin{aligned}
\mathbb{P}(\mathcal{E}_{n,N,\beta}^c) &= \mathbb{P}(\{o \notin \mathrm{int}[X_1, \ldots, X_N]\}) \\
&= \mathbb{P}_{\beta}^N(\{(x_1, \ldots, x_N) \in B_n^N : o \notin \mathrm{int}[x_1, \ldots, x_N]\}) \\
&\leq e^{-c(n,\beta)N}
\end{aligned}$$

for some constant $c(n,\beta)$ satisfying $0 < C_1 \leq c(n,\beta) \leq C_2$ where C_1, C_2 are absolute constants.

Step 3(i): Reduction to beta polytopes containing o

- By the law of total expectation,

$$\begin{aligned}
&\mathbb{E}[\mathrm{vol}_n(t_{n,N,\beta}B_n \triangle P^{\beta}_{n,N})] \\
&\qquad = \mathbb{E}[\mathrm{vol}_n(t_{n,N,\beta}B_n \triangle P^{\beta}_{n,N})|\mathcal{E}_{n,N,\beta}]\mathbb{P}(\mathcal{E}_{n,N,\beta}) \\
&\qquad + \mathbb{E}[\mathrm{vol}_n(t_{n,N,\beta}B_n \triangle P^{\beta}_{n,N})|\mathcal{E}^c_{n,N,\beta}]\mathbb{P}(\mathcal{E}^c_{n,N,\beta}) \\
&\qquad \le \mathbb{E}[\mathrm{vol}_n(t_{n,N,\beta}B_n \triangle P^{\beta}_{n,N})|\mathcal{E}_{n,N,\beta}] \\
&\qquad + \mathrm{vol}_n(B_n)e^{-c(n,\beta)N}.
\end{aligned}$$

- The second term is negligible and we shall henceforth ignore it.

Step 3(ii): Reduction to simplicial polytopes

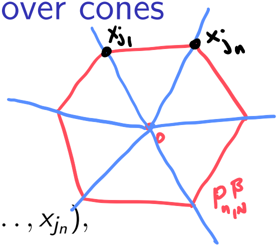
Let

$$E_{n,N,\beta} := \{(x_1,\dots,x_N) \in B_n^N : o \in \mathrm{int}[x_1,\dots,x_N] \\ \text{and } [x_1,\dots,x_n] \text{ is simplicial}\}.$$

Since $P_{n,N}^{\beta}$ is simplicial with probability 1,

$$\begin{aligned} &\mathbb{E}[\mathrm{vol}_n(t_{n,N,\beta}B_n \triangle P_{n,N}^{\beta})|\mathcal{E}_{n,N,\beta}] \\ &= 2\int_{B_n}\cdots\int_{B_n} \mathrm{vol}_n(t_{n,N,\beta}B_n \setminus [x_1,\dots,x_N])\mathbb{1}_{E_{n,N,\beta}}(x_1,\dots,x_N) \\ &\qquad \times \mathrm{d}\mathbb{P}_\beta(x_1)\cdots \mathrm{d}\mathbb{P}_\beta(x_N). \end{aligned}$$

Step 3(iii): Express the integral as a sum over cones



For $(x_1, \dots, x_N) \in E_{n,N,\beta}$, we have

$$\mathbb{R}^n = \bigcup_{[x_{j_1},\dots,x_{j_n}] \in \mathcal{F}_{n-1}([x_1,\dots,x_N])} \mathrm{cone}(x_{j_1}, \dots, x_{j_n}),$$

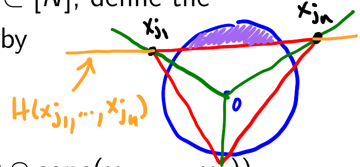
where $\mathcal{F}_{n-1}([x_1, \dots, x_N])$ is the set of facets of $[x_1, \dots, x_N]$ and

$$\mathrm{cone}(y_1, \dots, y_m) := \left\{ \sum_{i=1}^{m} a_i y_i \, : \, a_i \geq 0, \, i \in [m] \right\}$$

denotes the cone spanned by $y_1, \dots, y_m \in \mathbb{R}^n$.

Step 3(iii): Express the integral as a sum over cones

- For $y_1, \dots, y_n \in \mathbb{R}^n$ whose affine hull is an $(n-1)$-dimensional hyperplane $H(y_1, \dots, y_n)$, let $H^+(y_1, \dots, y_n)$ denote the halfspace with $o \in H^+(y_1, \dots, y_n)$.
- For $x_1, \dots, x_N \in \mathbb{R}^n$ and $\{j_1, \dots, j_n\} \subset [N]$, define the functional $\Phi^{\beta}_{j_1,\dots,j_n} : (\mathbb{R}^n)^N \to [0, \infty)$ by



$$\Phi^{\beta}_{j_1,\dots,j_n}(x_1, \dots, x_N)$$
$$:= \mathrm{vol}_n(t_{n,N,\beta} B_n \cap H^-(x_{j_1}, \dots, x_{j_n}) \cap \mathrm{cone}(x_{j_1}, \dots, x_{j_n})),$$

if $o \in \mathrm{int}([x_1, \dots, x_N])$ and $\dim([x_{j_1}, \dots, x_{j_n}]) = n - 1$; otherwise, set $\Phi^{\beta}_{j_1,\dots,j_n}(x_1, \dots, x_N) := 0$.

Step 3(iii): Express the integral as a sum over cones

Then

$$
\begin{aligned}
&\mathbb{E}[\mathrm{vol}_n(t_{n,N,\beta} B_n \triangle P^{\beta}_{n,N}) | \mathcal{E}_{n,N,\beta}] \\
&= 2 \int_{B_n} \cdots \int_{B_n} \mathrm{vol}_n(t_{n,N,\beta} B_n \setminus [x_1, \dots, x_N]) \mathbb{1}_{E_{n,N,\beta}}(x_1, \dots, x_N) \\
&\qquad\qquad\qquad\qquad\qquad \times \mathrm{d}\mathbb{P}_\beta(x_1) \cdots \mathrm{d}\mathbb{P}_\beta(x_N) \\
&= 2 \int_{B_n} \cdots \int_{B_n} \sum_{\{j_1,\dots,j_n\} \subset [N]} \Phi^{\beta}_{j_1,\dots,j_n}(x_1, \dots, x_N) \, \mathrm{d}\mathbb{P}_\beta(x_1) \cdots \mathrm{d}\mathbb{P}_\beta(x_N) \\
&= 2 \binom{N}{n} \int_{B_n} \cdots \int_{B_n} \Phi^{\beta}_{1,\dots,n}(x_1, \dots, x_N) \, \mathrm{d}\mathbb{P}_\beta(x_1) \cdots \mathrm{d}\mathbb{P}_\beta(x_N).
\end{aligned}
$$

Step 3(iv): Apply the affine Blaschke-Petkanschin formula

We apply the affine Blaschke–Petkantschin formula to derive

$$\mathbb{E}[\mathrm{vol}_n(t_{n,N,\beta} B_n \triangle P_{n,N}^{\beta}) | \mathcal{E}_{n,N,\beta}]$$
$$= 2\omega_n (n-1)! \binom{N}{n} \int_{\mathbb{S}^{n-1}} \int_0^1 \int_{B_n \cap H} \cdots \int_{B_n \cap H}$$
$$\left[\int_{B_n} \cdots \int_{B_n} \Phi_{1,\dots,n}^{\beta}(x_1, \dots, x_N) \, \mathrm{d}\mathbb{P}_\beta(x_{n+1}) \cdots \mathrm{d}\mathbb{P}_\beta(x_N) \right] \times$$
$$\times \mathrm{vol}_{n-1}([x_1, \dots, x_n]) \, \mathrm{d}\mathbb{P}_\beta^H(x_1) \cdots \mathrm{d}\mathbb{P}_\beta^H(x_n) \, \mathrm{d}h \, \mathrm{d}\sigma(u).$$

After some standard computations...

...we finally arrive at the estimate

$$\mathbb{E}[\mathrm{vol}_n(t_{n,N,\beta}B_n \triangle P_{n,N}^{\beta})] \leq \left(1+\frac{1}{n+2\beta+3}\right) \frac{2n\omega_n}{n+2\beta+1} \frac{1}{N^n}\binom{N}{n}\left(\frac{d_{n,\beta}}{N}\right)^{\frac{2}{n+2\beta+1}} \times \int_0^{N\varphi_\beta(\frac{n+2\beta+2}{n+2\beta+3})} t^{n-1+\frac{2}{n+2\beta+1}}\left(1-\frac{t}{N}\right)^{N-n} \mathrm{d}t + e^{-C(n,\beta)N}.$$

Here $d_{n,\beta}$ is a specifically known constant depending on the second moment of a random beta simplex, $C(n,\beta)>0$, and $\varphi_\beta(h) = 1 - F_{1,\beta+\frac{n-1}{2}}(h)$ where $F_{1,\beta}$ is the cdf of the one-dimensional beta distribution,

$$F_{1,\beta}(h) = c_{1,\beta}\int_{-1}^{h}(1-x^2)^{\beta}\,\mathrm{d}x.$$

Conclusion of Step 3

By standard estimates we get that for large enough N,

$$\int_0^{N\varphi_\beta\left(\frac{n+2\beta+2}{n+2\beta+3}\right)} t^{n-1+\frac{2}{n+2\beta+1}} \left(1-\frac{t}{N}\right)^{N-n} \mathrm{d}t \leq (1+e^{-O(n+2\beta+1)})\Gamma\left(n+\frac{2}{n+2\beta+1}\right).$$

Thus for large enough N,

$$\mathbb{E}[\mathrm{vol}_n(t_{n,N,\beta}B_n \triangle P_{n,N}^\beta)] \leq \left(1+O\left((n+2\beta+3)^{-1}\right)\right)\frac{2\omega_n}{n+2\beta+1} \times \frac{n!}{N^n}\binom{N}{n}\frac{\Gamma\left(n+\frac{2}{n+2\beta+1}\right)}{\Gamma(n)}\left(\frac{d_{n,\beta}}{N}\right)^{\frac{2}{n+2\beta+1}} + e^{-C(n,\beta)N}.$$

Estimates for the gamma function yield
$\mathbb{E}[\mathrm{vol}_n(B_n \triangle t_{n,N,\beta}^{-1}P_{n,N}^\beta)] \leq \frac{Cn\,\mathrm{vol}_n(B_n)}{n+2\beta+1}N^{-\frac{2}{n+2\beta+1}}$.

Step 4: Going from local estimates to the global estimate

- Replace n by j.
- Select the parameter $\beta = \frac{n-j-2}{2}$, which corresponds to a j-dimensional projection of the uniform distribution on $\mathbb{S}^{n-1}$ ($\beta = -1$).
- Choose the scaling factor $t_{j,N,\frac{n-j-2}{2}}$.
- Substitute everything into the local estimate from Step 3.

Using the identity $V_j(B_n) = \begin{bmatrix} n \\ j \end{bmatrix} \mathrm{vol}_j(B_j)$, for large N we get

$$
\begin{aligned}
\mathbb{E}[\delta_j(t^{-1}_{j,N,\frac{n-j-2}{2}} P^{\mathrm{unif}}_{n,N}, B_n)] &= \begin{bmatrix} n \\ j \end{bmatrix} \mathbb{E}[\mathrm{vol}_j(t^{-1}_{j,N,\frac{n-j-2}{2}} P^{\beta=\frac{n-j-2}{2}}_{j,N} \triangle B_j)] \\
&\leq \begin{bmatrix} n \\ j \end{bmatrix} \frac{Cj\,\mathrm{vol}_j(B_j)}{n-1} N^{-\frac{2}{n-1}} \\
&= \frac{Cj}{n-1} V_j(B_n) N^{-\frac{2}{n-1}}.
\end{aligned}
$$

$\square$

Thank you!

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