Talagrand's Selector Process Conjecture, Suprema of Positive Empirical Processes & The Kahn-Kalai Conjecture

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Joint with Jinyoung Park

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I/ Suprema of Stochastic Processes & Talagrand's Conjecture

II/ The Kahn-Kalai Conjecture

III/ Proof Ideas

I/ Suprema of stochastic processes and Talagrand's conjecture:

* Gaussian Processes:

Let $T \subseteq \mathbb{R}^M$, $g \sim \mathcal{N}(0, I_M)$. Define $Z_t = g \cdot t$.

Theorem (Talagrand): $\exists C > 0$: The event $\mathcal{E} = \left\{ g : \sup_{t \in T} Z_t > C \, \mathbb{E} \sup_{t \in T} Z_t \right\}$

can be covered by a union of half-spaces

$$\mathcal{E} \subseteq \bigcup_{k=1}^{\infty} \left\{ g \cdot s_k \geq v_k \right\}$$

with

$$\sum_{k=1}^{\infty} \mathbb{P}(g \cdot s_k \geq v_k) < \frac{1}{2}.$$

Corollary of Talagrand's majorizing measure theorem.

Theme: We have extreme events, and would like to give explicit witnesses (covering by half-spaces) of these events.

* Selector Processes:

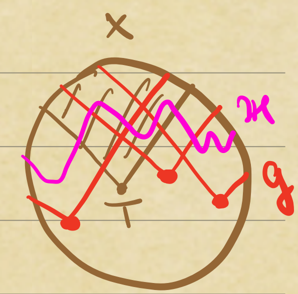
Let X be a finite set. Let Λ be a set of sequences $(\lambda_i)_{i \in X}$.

For a subset $W \sim X_p$ of X, we define the selector process

$$Z_\lambda = \lambda . W = \sum_{i \in W} \lambda_i \quad (\text{for } \lambda \in \Lambda),$$

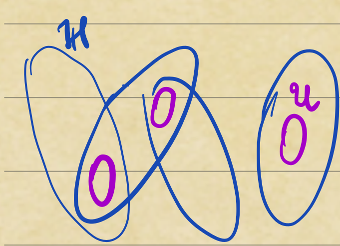
$$Z_\Lambda = \sup_{\lambda \in \Lambda} Z_\lambda .$$

(Positive selector process: $\lambda_i \geq 0 \ \forall \ \lambda \in \Lambda$)



Def: For $T \subseteq X$, let $\langle T \rangle = \{ W \subseteq X : T \subseteq W \}$.

(up-set / half-space)



For $\mathcal{H} \subseteq 2^X$, say that $\mathcal{U} \subseteq 2^X$ forms a cover for $\mathcal{H}$ if

$$\mathcal{H} \subseteq \bigcup_{T \in \mathcal{U}} \langle T \rangle .$$

$\mathcal{H}$ is p-small if there exists a cover $\mathcal{U} \subseteq 2^X$ with $\text{cost}(\mathcal{U}) := \sum_{T \in \mathcal{U}} p^{|T|} < \frac{1}{2}$.

Conj (Talagrand): $\exists C>0$: For any positive selector process ($\lambda_i \geq 0$ $\forall \lambda \in \Lambda$)

the event $\mathcal{E} = \left\{ W : \sup_{\lambda \in \Lambda} \lambda . W > C\, \mathbb{E} \sup_{\lambda \in \Lambda} \lambda . X_p \right\}$ is p-small,

i.e. there exists "half-spaces" $\langle T_k \rangle$ such that

$$\cdot\ \mathcal{E} \subseteq \bigcup_k \langle T_k \rangle,$$

and $$\cdot\ \sum_k p^{|T_k|} < \frac{1}{2}.$$

- $\mathbb{P}_p(\langle T_k \rangle) = p^{|T_k|}$.

The fractional version of the conjecture, that $\mathcal{E}$ is "fractionally p-small", is also open until our work.

* Empirical Processes:

Let $Y_1, \dots, Y_N \underset{iid}{\sim} \mathbb{P}$ on Π. Let $\mathcal{F}$ be a class of functions on Π

The empirical process is defined as $Z_f = \frac{1}{N} \sum_{i=1}^{N} f(Y_i)$, for $f \in \mathcal{F}$.

Again we are interested in $\sup_{f \in \mathcal{F}} Z_f$.

Conj (Talagrand): $\exists C > 0$: For any positive empirical process ($f \geq 0 \ \forall f \in \mathcal{F}$)

the event $\mathcal{E} = \left\{ \sup_f Z_f > C \, \mathbb{E} \sup_f Z_f \right\}$ can be covered by a

collection of half-spaces of the empirical measure $\mathcal{H}_{g,t} = \left\{ Z_g \geq t \right\}$ for $(g,t) \in \mathcal{C}$

such that $\sum_{(g,t) \in \mathcal{C}} \mathbb{P}(\mathcal{H}_{g,t}) < \frac{1}{2}$.

Motivation for the positive assumption:

Thm (Generalized Bernoulli Conjecture, Informal): Let $X_1, \dots, X_n \underset{iid}{\sim} Ber(p)$,

$\Lambda = \{(\lambda_1, \dots, \lambda_n)\}$, then $\sup_{\lambda \in \Lambda} \left| \sum_{i=1}^{n} \lambda_i (X_i - p) \right|$ is characterized by a

metric-geometric quantity, and the suprema of a positive selector process.

Thm (Fundamental Thm of Empirical Processes): Let $Y_1, \dots, Y_N \underset{iid}{\sim} P$ on Π.
Let $\mathcal{F}$ be a class of mean-zero functions on Π.
Then $\sup_{f \in \mathcal{F}} |Z_f| = \sup_{f \in \mathcal{F}} \left| \frac{1}{N} \sum_{i=1}^{N} f(Y_i) \right|$ is characterized by a metric-geometric

quantity, and the suprema of a positive empirical process.

«Chaining explains all the boundedness due to cancellation,
but what could we ask about boundedness of processes where no cancellation occurs?»
(Talagrand, Chap 13 "Unfulfilled dreams")

Thm (Generalized Bernoulli Conjecture): Let $X_i \sim \mathrm{Ber}(p)$, and $Z_\lambda = \sum_{i \le N} \lambda_i (X_i - p)$, and $M = \mathbb{E} \sup_{\lambda \in \Lambda} |Z_\lambda|$.

We can write $\Lambda \subset \Lambda_1 + \Lambda_2$ and

$$\gamma_2(\Lambda_1, d_2) \le \frac{CM}{\sqrt{p}}, \qquad \gamma_1(\Lambda_1, d_\infty) \le CM,$$

$$\mathbb{E} \sup_{\lambda_2 \in \Lambda_2} \sum_{i \le N} X_i |\lambda_{2,i}| \le CM.$$

Furthermore, if $\Lambda \subset \Lambda_1 + \Lambda_2$, then

$$Z_\Lambda \le C\sqrt{p}\,\gamma_2(\Lambda_1, d_2) + C\gamma_1(\Lambda_1, d_\infty) + L\,\mathbb{E} \sup_{\lambda_2 \in \Lambda_2} \sum_{i \le N} X_i |\lambda_{2,i}|.$$

Thm (Fundamental Thm of Empirical Processes): For $\mathcal{F} \subseteq L^2(P)$, $P(f) = 0$ for $f \in \mathcal{F}$, let $M = \mathbb{E} \sup_f Z_f$.

There exists $\mathcal{F}_1, \mathcal{F}_2$ with $\mathcal{F} \subseteq \mathcal{F}_1 + \mathcal{F}_2$,

$$\gamma_2(\mathcal{F}_1, d_2) \leq C\sqrt{N}\, M,$$

$$\gamma_1(\mathcal{F}_1, d_\infty) \leq CNM,$$

and

$$\mathbb{E} \sup_{f_2 \in \mathcal{F}_2} \frac{1}{N} \sum_{i=1}^{N} |f_2(Y_i)| \leq CM.$$

Furthermore, for any such decomposition,

$$\mathbb{E} \sup_f Z_f \leq \frac{C}{\sqrt{N}} \gamma_2(\mathcal{F}_1, d_2) + \frac{C}{N} \gamma_1(\mathcal{F}_1, d_\infty) + C\,\mathbb{E} \sup_{\mathcal{F}_2} Z_{|f_2|}.$$

Thm (Park-P.): $\exists C>0$: For any positive selector process ($\lambda_i \geq 0 \ \forall \lambda \in \Lambda$)

the event $\mathcal{E} = \left\{ W : \sup_{\lambda\in\Lambda} \lambda . W > C \, \mathbb{E} \sup_{\lambda \in \Lambda} \lambda . X_p \right\}$ is p-small,

i.e. there exists "half-spaces" $\langle T_k \rangle$ such that

$$\cdot \ \mathcal{E} \subseteq \bigcup_k \langle T_k \rangle,$$

and $$\cdot \ \sum_k p^{|T_k|} < \frac{1}{2}.$$

Thm (Park-P.): $\exists C>0$: For any positive empirical process ($f \geq 0 \ \forall f \in \mathcal{F}$)

the event $\mathcal{E} = \left\{ \sup_f Z_f > C \, \mathbb{E} \sup_f Z_f \right\}$ can be covered by a

collection of half-spaces of the empirical measure $\mathcal{H}_{g,t} = \{ Z_g \geq t \}$ for $(g,t) \in \mathcal{C}$

such that $$\sum_{(g,t)\in\mathcal{C}} \mathbb{P}(\mathcal{H}_{g,t}) < \frac{1}{2}.$$

Theorem: $\exists C>0$: For any positive selector process ($\lambda_i \geq 0 \ \forall \lambda \in \Lambda$)

the event $\mathcal{E} = \left\{ \sup_{\lambda\in\Lambda} \lambda \cdot W \quad > C \, \mathbb{E} \sup_{\lambda\in\Lambda} \lambda \cdot X_p \right\}$ is p-small,

i.e. there exists "half-spaces" $\langle T_k \rangle$ such that

$$\cdot \quad \mathcal{E} \subseteq \bigcup_k \langle T_k \rangle,$$

and
$$\cdot \quad \sum_k p^{|T_k|} < \frac{1}{2}.$$

Equivalent formulation:

Theorem: There exists $C>0$ such that the following holds:

Let $\mathcal{H} \subseteq 2^X$ be not p-small.

Assume that each $H \in \mathcal{H}$ is associated weights $\lambda^H : X \to \mathbb{R}_{\geq 0}$

such that λ^H is supported on H, and $\sum_{i\in X} \lambda_i^H \geq 1$.

Then $\mathbb{E} \sup_{H\in\mathcal{H}} \lambda^H \cdot X_p \quad \geq \frac{1}{C}.$

Theorem: There exists $C > 0$ such that the following holds:
Let $\mathcal{H} \subseteq 2^X$ be not p-small.
Assume that each $H \in \mathcal{H}$ is associated some weights $\lambda^H : X \to \mathbb{R}_{\geq 0}$
such that λ^H is supported on H, and $\sum_{i \in X} \lambda_i^H \geq 1$.

Then $$\mathbb{E} \sup_{H \in \mathcal{H}} \lambda^H \cdot X_p \geq \frac{1}{C}.$$

Corollary: $|H| = \ell \ \forall H \in \mathcal{H}$, and $\lambda_i^H = \frac{1}{\ell} \mathbb{1}(i \in H)$.

Then if $\mathcal{H}$ is not p-small, whp, there is $H \in \mathcal{H}$ with $|X_p \cap H| \geq \frac{|H|}{C}$.

A slight strengthening is the key lemma in the proof of the Kahn-Kalai Conjecture.

I/ The Kahn-Kalai Conjecture:

1) Thresholds:

- Let X be a finite set,
- X_p a p-random subset of X.
- $\mathcal{G}$ increasing property (of subsets of X): $\mathcal{G} \subseteq 2^X$

  W satisfies the property if $W \in \mathcal{G}$.

  $W' \supseteq W$, $W \in \mathcal{G} \Rightarrow W' \in \mathcal{G}$

$X = \binom{[n]}{2}$,

$X_p \longleftrightarrow G(n,p)$ Erdős-Rényi

$\mathcal{G}$ increasing graph property

E.g.: connected, having perfect matching, ...

Main Example: (Graph containment) $\mathcal{G}_H = \{ W : W \text{ contains a copy of } H \}$.

H can be a fixed graph ($H = \triangle$), or a graph depending on n (H = Hamiltonian cycle)

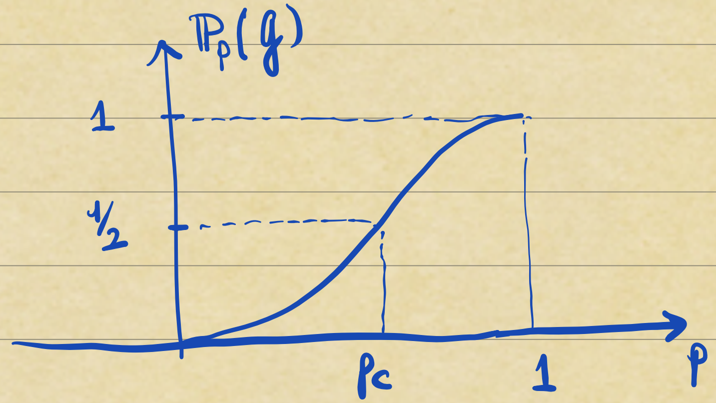
Rmk: Any increasing property $\mathcal{G}$ can be described as

$$\mathcal{G} = \mathcal{G}_{\mathcal{H}} = \{ W : W \supseteq H \text{ for some } H \in \mathcal{H} \}$$

for some family $\mathcal{H} \subseteq 2^X$.

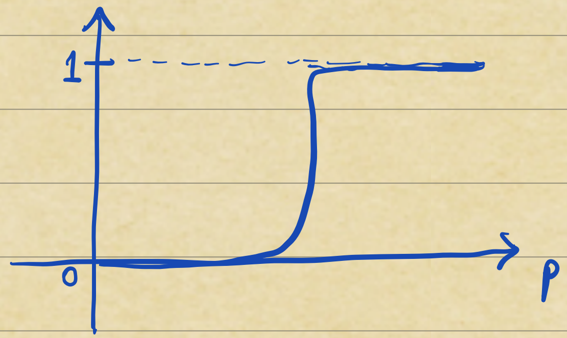
Main interest: What is $\mathbb{P}_p(\mathcal{G}) = \mathbb{P}_p(X_p \in \mathcal{G})$

$$= \mathbb{P}_p(X_p \supseteq H \text{ for some } H \in \mathcal{H})$$



Def: The threshold $p_c(\mathcal{G})$ is the unique $p \in [0,1]$ such that

$$\mathbb{P}_p(\mathcal{G}) = \frac{1}{2}.$$



"Sharp Threshold"
(Friedgut, Kahn - Kalai - Linial, ...)

Main question: Location of threshold $p_c(\mathcal{G})$.

2) Expectation threshold:

If $\mathbb{P}_p(X_p \supseteq H \text{ for some } H \in \mathcal{H}) \geq \frac{1}{2}$, then

$$\mathbb{E}_p\Big[\sum_{H \in \mathcal{H}} \mathbb{1}(H \subseteq X_p)\Big] \geq \frac{1}{2}$$

$$\sum_{H \in \mathcal{H}} p^{|H|} = \operatorname{cost}_p(\mathcal{H})$$

Hence if $\sum_{H \in \mathcal{H}} p^{|H|} < \frac{1}{2}$, then

$\mathbb{P}_p(X_p \supseteq H \text{ for some } H \in \mathcal{H}) < \frac{1}{2}$, so $p \leq p_c(\mathcal{G})$.

Similarly, for any cover $\mathcal{H}'$ of $\mathcal{H}$ (any $H \in \mathcal{H}$ contains some $H' \in \mathcal{H}'$):

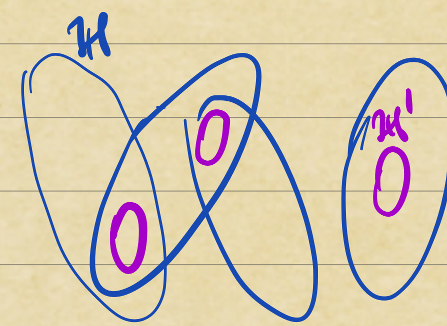
if $\mathbb{P}_p(X_p \supseteq H' \text{ for some } H' \in \mathcal{H}') \geq \mathbb{P}_p(X_p \supseteq H \text{ for some } H \in \mathcal{H}) \geq \frac{1}{2}$,

then $\sum_{H' \in \mathcal{H}'} p^{|H'|} \geq \frac{1}{2}$. So if $\sum_{H' \in \mathcal{H}'} p^{|H'|} < \frac{1}{2}$, then $p \leq p_c(\mathcal{G})$.

Def: The expectation threshold $p_E(\mathcal{G})$ is the largest p s.t. $\mathcal{H}$ is p-small:

There exists a cover $\mathcal{H}'$ of $\mathcal{H}$ with $\sum_{H' \in \mathcal{H}'} p^{|H'|} < \frac{1}{2}$.

Cor: $p_c(\mathcal{G}) \geq p_E(\mathcal{G})$.



* It is often easy to determine $p_E(\mathcal{G})$ up to constant factor.

3) The Kahn-Kalai Conjecture:

Conj: There exists $C > 0$ s.t. for every increasing property $\mathcal{G}$,

$$\left(p_E(\mathcal{G}) \leq \right) \; p_c(\mathcal{G}) \leq C \, p_E(\mathcal{G}) \log |X| .$$

Thm (Park-P.): For all $\mathcal{H} \subseteq 2^X$ with $|H| \leq \ell \;\; \forall H \in \mathcal{H}$,

if $\mathcal{H}$ is not p-small (any cover $\mathcal{H}'$ of $\mathcal{H}$ has $\mathrm{cost}(\mathcal{H}') = \sum_{H' \in \mathcal{H}'} p^{|H'|} \geq \frac{1}{2}$)

then $$\mathbb{P}\left(X_{Cp\log \ell} \supseteq H \text{ for some } H \in \mathcal{H}\right) \geq 1 - o_{\ell, C}(1).$$

Fractional version $\begin{pmatrix} \text{Alweiss - Lovett - Wu - Zhang} \\ \text{Frankston - Kahn - Narayanan - Park} \end{pmatrix}$

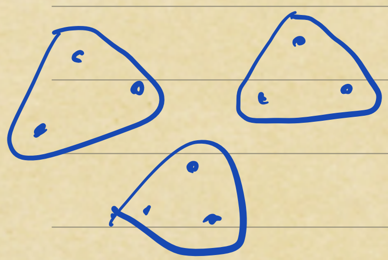
4) Applications:

E.g. 1: Threshold for containing a bounded degree spanning tree (Kahn's conjecture)

$$p_E(\mathcal{G}_T) \le p_{fE}(\mathcal{G}_T) \le O(1/n) \Rightarrow p_c(\mathcal{G}_T) \le O\left(\frac{\log n}{n}\right), \text{ tight}$$

[Montgomery 2019]

E.g. 2: Shamir's problem on hypergraph matching:



In r-uniform hypergraph:

$$p_E(\mathcal{G}_{r\text{-pm}}) \le O(n^{1-r})$$

$$\Rightarrow p_c(\mathcal{G}_{r\text{-pm}}) \le O((\log n) n^{1-r}), \text{ tight.}$$

[Johansson - Kahn - Vu 2008]

III/ Proof Ideas:

Theorem: There exists $C > 0$ such that the following holds:

Let $\mathcal{H} \subseteq 2^X$ be not p-small.

Assume that each $H \in \mathcal{H}$ is associated some weights $\lambda^H : X \to \mathbb{R}_{\geq 0}$ such that λ^H is supported on H, and $\sum_{i \in X} \lambda_i^H \geq 1$.

Then
$$\mathbb{E} \sup_{H \in \mathcal{H}} \lambda^H \cdot X_p \geq \frac{1}{C}.$$

Theorem: Let $\mathcal{H} \subseteq 2^X$ with $|H| \leq \ell$ $\forall H \in \mathcal{H}$ be not p-small.

Then
$$\mathbb{P}(X_{Cp\log \ell} \supseteq H \text{ for some } H \in \mathcal{H}) \geq 1 - o_{\ell, C}(1).$$

Plan: Instead of working with independently sampled elements, we can also work with a uniform random subset of size $Cp|X|$.

- Say that W is bad if $\sup_{H \in \mathcal{H}} \lambda^H \cdot W < \frac{1}{C}$.
- For each W bad, we construct a cover $\mathcal{U}(W)$ of $\mathcal{H}$.
- ☆ We show that $\mathbb{E}_{W \sim X_{Cp}} \left[\mathbb{1}(W \text{ bad}) \operatorname{cost}(\mathcal{U}(W)) \right] < \frac{1}{C}$.
- Since $\operatorname{cost}(\mathcal{U}(W)) \geq \frac{1}{2}$, $\mathbb{P}(X_{Cp} \text{ bad}) < \frac{2}{C}$.

$$\Rightarrow \mathbb{E} \sup_{H} \lambda^H \cdot X_p \geq \left(1 - \frac{2}{C}\right) \frac{1}{C^2}.$$

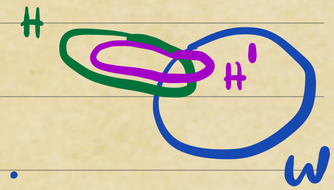
We will see how to prove ☆ in increasing generality:

Special Case: $\lambda_i^H = \frac{1}{\ell}$ for $i \in H$. <Main lemma of Kahn-Kalai Conj>

General Setting

Special Case : $\lambda_i^H = \frac{1}{\ell} \mathbb{1}(i \in H)$.

We prove a slightly stronger result which would be the main lemma in the proof of the kahn-kalai conjecture.



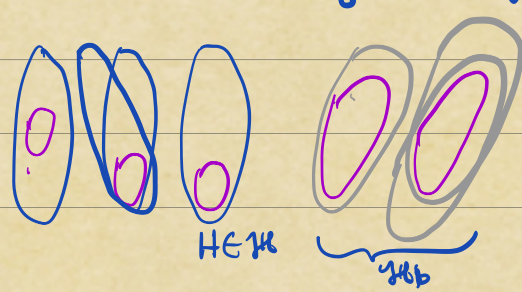
Note that $\lambda^H \cdot W = \frac{|W \cap H|}{|H|}$.

Say that W is H-bad if $\sup_{\substack{H' \in \mathcal{H}: \\ H' \subseteq W \cup H}} \frac{|W \cap H'|}{|H'|} < \frac{1}{C}$.

Lemma: For each W, let $\mathcal{H}_b(W)$ be the collection of $H \in \mathcal{H}$ where W is H-bad. There is a cover $\mathcal{U}(W)$ of $\mathcal{H}_b(W)$ with $\mathbb{E}_W[\text{cost}(\mathcal{U}(W))] < \frac{1}{C}$.

Obs 1. If W is bad, then $\mathcal{H}_b(W) = \mathcal{H}$, so $\mathcal{U}(W)$ is a cover of $\mathcal{H}$. So the lemma implies ☆ in this case.

Obs 2. The kahn-kalai conjecture follows from iterating the lemma.

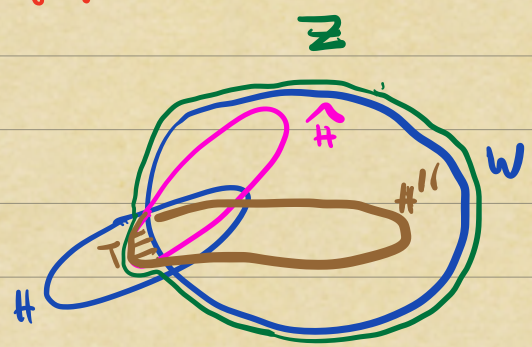


If $H \notin \mathcal{H}_b(W)$, can replace H by subset $\psi^W(H)$ s.t. $|\psi^W(H)| \leq (1 - \frac{1}{C})\ell$

$W \cup \psi^W(H) \supseteq H' \in \mathcal{H}$.

Insight: Find the most "structured" part of H that can be efficiently encoded based on interaction of H and W

Proof of Lemma:



Def: Let $\hat{H} \in \mathcal{H}$ be so that $\hat{H} \subseteq H \cup W$ and $|\hat{H} \setminus W|$ is minimal.

Let $T = T(H,W) = \hat{H} \setminus W$. Let $Z = W \cup T$.

minimum fragment

Key Property: For all $H'' \in \mathcal{H}$, $H'' \subseteq Z$, we have $T \subseteq H''$.

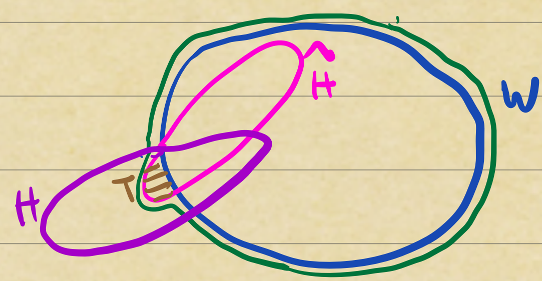
Pf: If $H'' \subseteq Z$, $T \not\subseteq H''$, then $H'' \subseteq H \cup W$ and $|H'' \setminus W| < |T|$, contradict minimality

Construct cover $\mathcal{U}(W) = \{ T(H,W) : H \in \mathcal{H}(W),\ |T(H,W)| \geq .9\ell \}$

$(\mathcal{H}_b(W) = \{ H,\ |T(H,W)| \geq .9\ell \})$

Key Lemma <-----> Few (W,T): T is a large minimum fragment of W.
$\Updownarrow$
Few (Z,T): T is a large subset of $\tilde{H}(Z)$.

$$\mathbb{E}[\mathrm{Cost}(\mathcal{U}(W))] = \frac{1}{\binom{|X|}{Cp|X|}} \sum_{t \geq 0.9\ell} p^t \left|\{(W,T) : T = T(H,W) \text{ for some } H \in \mathcal{H}(W),\ |T| = t\}\right|$$



For each Z with $|Z| = Cp|X| + t$, define $\tilde{H} = \tilde{H}(Z)$ as any set $\tilde{H} \in \mathcal{H}$ with $\tilde{H} \subseteq Z$.

Claim: $\left|\{(W,T) : T = T(H,W) \text{ for some } H \in \mathcal{H}(W),\ |T| = t\}\right|$
$$\leq \left|\{(Z,T) : |Z| = Cp|X| + t,\ T \subseteq \tilde{H}(Z)\}\right|$$

Proof: The key property implies that for $Z = Z(H,W)$, $T = T(H,W)$, any $\tilde{H} \subseteq Z$ contains T.
We can recover W from (Z,T) as $Z \setminus T$.

$$\mathbb{E}[\mathrm{Cost}(\mathcal{U}(W))] \leq \frac{1}{\binom{|X|}{Cp|X|}} \sum_{t \geq .9\ell} p^t \cdot \underbrace{\binom{|X|}{Cp|X|+t}}_{\text{choice of } Z} \cdot \underbrace{2^\ell}_{\text{choice of } T \subseteq \tilde{S}(Z)} \leq \sum_{t \geq .9\ell} p^t (Cp)^{-t} 2^\ell \leq \tilde{C}^{-\ell}.$$

General Case:

Reductions:

We can assume that: $x_i^H \in \{1, 100^{-1}, 100^{-2}, \dots, 100^{-\log|X|}\}$.
Let h_j be the number of i with $x_i^H = 100^{-j}$, and H_j the set of such i.

Call $\underline{h} = (h_0, h_1, h_2, \dots)$ the profile of H.
We can wlog work with sets in $\mathcal{H}$ with a fixed profile.

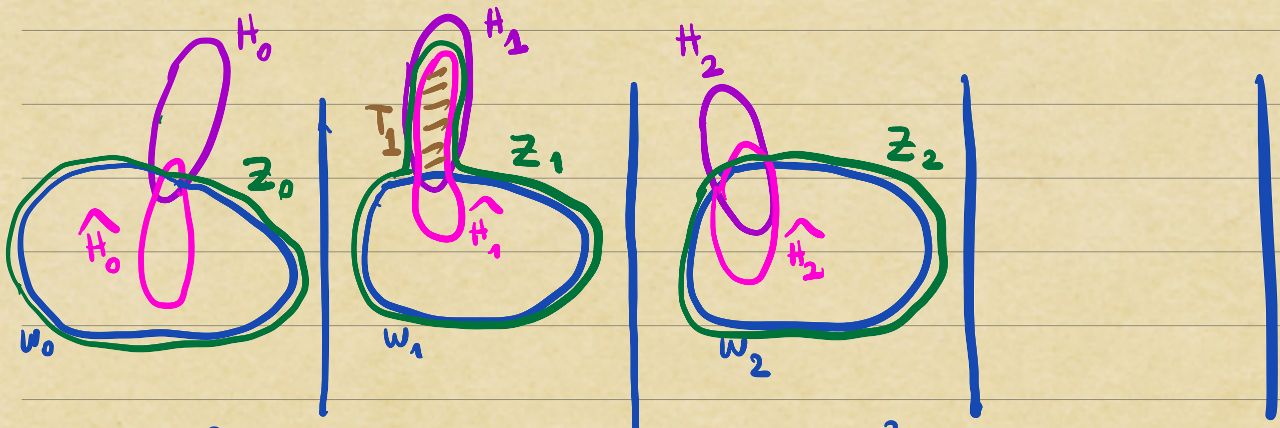
Goal: For each H and bad W, find $T \subseteq H$ of size t, such that for $Z = W \cup T$, and an appropriate choice $\tilde{H}(Z)$ (depending only on Z), we can guarantee:

We can point out an explicit subset of $\tilde{H}(Z)$ with size $O(t)$ that is guaranteed to contain T.

Difficulty 1: "Size does not capture weight".

$\hookrightarrow$ zoom in and only use information from H in useful parts

$\lambda_i^H = \lambda_i$



$\lambda_i = 100^{-0}$ $\lambda_i = 100^{-1}$ $\lambda_i = 100^{-2}$ ...

$b_0 = 0$ $b_1 = 1$ $b_2 = 0$

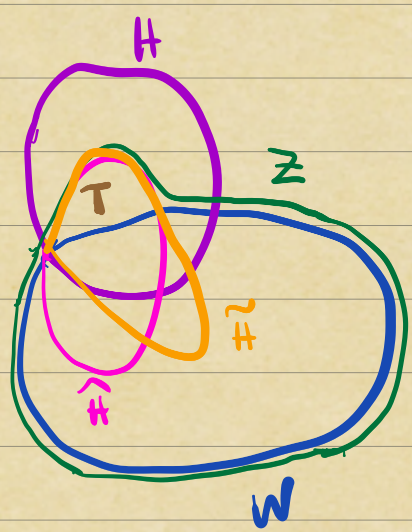
- Where $b_j = 0$, $|\hat{H}_j \cap W_j| \geq c h_j$, encode $Z_j = W_j$.
- Where $b_j = 1$, encode $Z_j = W_j \cup T_j$. ($\sum_{b_j=1} |T_j|$ minimal)

- For appropriate b, $\sum_{b_j=1} |T_j| \geq c \sum_{b_j=1} h_j$.
- T is a subset of $\bigcup_{b_j=1} \hat{H}_j$.

Difficulty 2: "Weights move" $\longrightarrow$ motivates minimal fragments!

Again, given H and W, we find $T \subseteq H$, $Z = T \cup W$ and $b \in \{0,1\}^{\log N}$ such that there is $\hat{H}$ with:

- The weight of $\hat{H}_j$ in Z is large if $b_j = 0$,
- $\hat{H}_j \subseteq Z$ if $b_j = 1$.



Significant challenge: weight of elements vary based on the set, no canonical partition.

guarantee T lies inside correct part of $\tilde{H}$.

Def: $B \le \log N$ minimal: $T \subseteq H$ is minimal such that for $Z = W \cup T$, we can find $\hat{H}$:

$$\hat{H}_j \subseteq Z \quad \text{for } j \le B,$$

$$\sum_{j>B} \sum_{i \in Z \cap \hat{H}_j} \lambda^{\hat{H}}(i) \geq c\left(\sum_{j>B} \sum_{i \in \hat{H}_j} \lambda^{\hat{H}}(i) - 100^{-B}\Big(\sum_{j \le B} h_j - |T| \Big) \right).$$

Conj (Talagrand): Let γ_n be the standard Gaussian distribution in $\mathbb{R}^n$, for any symmetric compact $A \subseteq \mathbb{R}^n$ with $\gamma_n(A) \geq 1 - \frac{1}{C}$, $CA = \underbrace{A + \dots + A}_{C \text{ times}}$ contains a convex compact set B with $\gamma_n(B) \geq \frac{1}{2}$.

Algorithmic versions?