About Bezout inequalities for mixed volumes

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Mixed volume : Minkowski's definition

Denote by $\mathcal{K}_n = \{K \subset \mathbb{R}^n : K \text{ compact convex set}\}$.

Let $K, L \in \mathcal{K}_n$. Then $Vol_n(\lambda K + \mu L)$ is a polynomial in (λ, μ) :

$$Vol_n(\lambda K + \mu L) = \sum_{k=0}^{n} \binom{n}{k} v_k \lambda^k \mu^{n-k}$$

where $v_k = V_n(K[k], L[n-k]) = V_n(K, ..., K, L, ..., L)$ are called mixed volumes.

Mixed volume : Minkowski's definition

- Let $K, L \in \mathcal{K}_n$. Then $Vol_n(\lambda K + \mu L) = \sum_{k=0}^{n} \binom{n}{k} v_k \lambda^k \mu^{n-k}$
- Let $K_1, ..., K_m \in \mathcal{K}_n$. Then :

$$Vol_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{\substack{a=(a_1,\ldots,a_m) \\ |a|=n}} \binom{n}{a} v_a \ \lambda^a$$

where $v_a = V_n(K_1[a_1], \ldots, K_m[a_m])$ are called mixed volumes.

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  where $v_a = V_n(K_1[a_1], \ldots, K_m[a_m])$ are called mixed volumes.
- $V_n : \mathcal{K}_n^n \to [0, +\infty)$ is a multilinear, continuous functional.

  Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transform. Then :

$$V_n(TK_1, ..., TK_n) = det(T) V_n(K_1, ..., K_n)$$

Bezout inequality

Let $f_1, ..., f_r : \mathbb{R}^n \to \mathbb{R}$ be polynomials. Denote by $X_1, ..., X_r$ the associated algebraic varieties $\quad (X_i := \{x \in \mathbb{R}^n : f_i(x) = 0\})$.

The *Bezout inequality* states that :

$$deg(X_1 \cap ... \cap X_r) \leq \prod deg(X_i) \qquad [B]$$

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We can reformulate [B] within the language of mixed volumes :

$$V(P_1, ..., P_r, \Delta[n-r])V(\Delta)^{r-1} \leq \prod_{i=1}^{r} V(P_i, \Delta[n-1])$$

thanks to a theorem by Bernstein, Kushnirenko and Khovanskii.

Bezout inequality (again)

Let $f_1, ..., f_n : \mathbb{R}^n \to \mathbb{R}$ be polynomials.
Let $X = X_2 \cap ... \cap X_n$ of dimension 1, and $Y = X_1$ (codim.1) .
Then Bezout inequality :

$$deg(X \cap Y) \leq deg(X)deg(Y) \qquad [B]$$

translates to

$$V_n(P_1, ..., P_n)V_n(\Delta) \leq V_n(P_2, ..., P_n, \Delta)V_n(P_1, \Delta[n-1]).$$

(which allows to recover previous inequality [B])

Relaxed Bezout inequality

- for the n-simplex Δ :

$$V(L_1, ..., L_n)V(\Delta) \leq V(L_2, ..., L_n, \Delta)V(L_1, \Delta[n-1]).$$

- Thanks to Diskant inequality, J. Xiao has shown (2019) :

$$V(L_1, ..., L_n)V(K) \leq nV(L_2, ..., L_n, K)V(L_1, K[n-1])$$

for any convex bodies $L_1, ..., L_n$, and for any K.

Bezout constants

We define :

$$b_2(K) = \max_{L_1,L_2} \frac{V(L_1, L_2, K[n-2])V(K)}{V(L_1, K[n-1])V(L_2, K[n-1])} \geq 1$$

And similarly

$$b(K) = \max_{L_1,\ldots,L_n} \frac{V(L_1, \ldots, L_n)V(K)}{V(L_2, \ldots, L_n, K)V(L_1, K[n-1])} \geq 1$$

So that :

- $b_2(\Delta) = b(\Delta) = 1$;
- $\forall K, 1 \leq b_2(K) \leq b(K)$;
- by [Diskant, Xiao] : $\max_K b(K) \leq n$.
- $\forall K$, $b(TK) = b(K)$, for any (full-rank) affine T.

Who are the minimizers ?

Question [SZ '15]

For which bodies do we have $b_2(K) = 1$?

Question [SSZ '18]

For which bodies do we have $b(K) = 1$?

SZ '15 $\rightarrow$ [Soprunov, Zvavitch] (2015)
SSZ '18 $\rightarrow$ [Saroglou, Soprunov, Zvavitch] (2018)

Who are the minimizers ?

Qstn [SZ '15] For which K, do we have $b_2(K) = 1$?

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- **Theorem**[SSZ '18] If $b(K) = 1$, then $K = \Delta$.

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- **Theorem**[SSZ '18] If $b(K) = 1$, then $K = \Delta$.
- this doesn't close former question, since $b_2(K) \leq b(K)$.

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- **Theorem**[SSZ '18] .If $b_2(P) = 1$, then $P = \Delta$.
  (where P is an n-polytope)

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  (where P is an n-polytope)
- **Prop**[SZ '15] if $b_2(K) = 1$, then $K \neq A + B$ (with $A \not\equiv B$)
  (K cannot be decomposable)

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- **Thm**[SSZ '18] Let $P \in \mathbf{Poly}_n$.Then $b_2(P) = 1 \Rightarrow P = \Delta$.
- **Thm**['15, '18] if $b_2(K) = 1$, then K cannot be *weakly decomposable* ($\to K \notin \mathcal{W}_n$)

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  $\to$ excludes bodies with (somewhere) smooth boundary.

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$\longrightarrow$ recovers characterization among polytopes, since $\mathbf{Poly}_n \cap \mathcal{W}_n = \mathbf{Poly}_n \setminus \{\Delta\}$.

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- **Thm**['15, '18] if $b_2(K) = 1$, then K cannot be *weakly decomposable* ($\to K \notin \mathcal{W}_n$)
- ... some more *restrictions*, eg : at most finitely many facets.

Qstn [SSZ '18] For which K do we have $b(K) = 1$?

- **Theorem**[SSZ '18] If $b(K) = 1$, then $K = \Delta$.

  $\to$ proof uses Wulff shape bodies, a pointwise Aleksandrov differentiation lemma, and builds on above *restrictions*.

A new necessary condition

Let $L \in \mathcal{K}_n$ be a k-dimensional. Denote :

$$Iso(L) := \frac{1}{k}\frac{Vol_{k-1}(\partial L)}{Vol_k(L)} =: \frac{1}{k}\frac{|\partial L|}{|L|}$$

Thm[S. 2022] If $b_2(K) = 1$, then :

For any facet F of K : $Iso(F) \leq Iso(K)$.

$\rightarrow$ recovers the "at most finitely many facets" restriction.

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Thm[S. 2022] If $b_2(K) = 1$, then, for any affine transform T :

For any facet F of K : $Iso(TF) \leq Iso(TK)$.

(since $b_2(K)$ is affine invariant, while $\max_F \frac{Iso(F)}{Iso(K)}$, is not)

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- **Question** : if $P \neq \Delta$, does there always exist

  an affine transform T s.t. $\max_F \frac{Iso(TF)}{Iso(TP)} > 1$?

... any questions ?

Thank you for your attention !!

Isoperimetric Inequalities for Hessian Valuations

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Ambient Spaces...

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$\mathrm{Conv}_{sc}(\mathbb{R}^n) := \{u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$: convex, l.s.c. and proper, $\lim_{|x|\to\infty} \frac{u(x)}{|x|} = +\infty\}$
with the topology induced by *epi*-convergence:
$u_j \to_{epi} u$ if

- For every sequence (x_j) that converges to x, $u(x) \leq \liminf_{j\to\infty} u_j(x_j)$.
- There exists a sequence (x_j) converging to x such that $u(x) = \lim_{j\to\infty} u_j(x_j)$.

...and their Valuations

Valuations on $\mathcal{K}^{n+1}$:

Functionals $Y : \mathcal{K}^{n+1} \to \mathbb{R}$ such that for every $K, L \in \mathcal{K}^{n+1}, K \cup L \in \mathcal{K}^{n+1}$

$$Y(K \cup L) + Y(K \cap L) = Y(K) + Y(L).$$

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Valuations on $\mathrm{Conv}_{sc}(\mathbb{R}^n)$:

Functionals $Z : \mathrm{Conv}_{sc}(\mathbb{R}^n) \to \mathbb{R}$ such that for every $u, v \in \mathrm{Conv}_{sc}(\mathbb{R}^n)$, $u \wedge v \in \mathrm{Conv}_{sc}(\mathbb{R}^n)$

$$Z(u \wedge v) + Z(u \vee v) = Z(u) + Z(v).$$

[Ludwig, Alesker, Colesanti, Mussnig, Knoerr...]

n-homogeneous valuations on $\mathcal{K}^{n+1}$

Theorem [McMullen, 1980]

A functional $Y : \mathcal{K}^{n+1} \to \mathbb{R}$ is a continuous, translation invariant real valued valuation which is n-homogeneous, if and only if there exists a continuous function $\eta : \mathbb{S}^n \to \mathbb{R}$ such that

$$Y(K) = \int_{\mathbb{S}^n} \eta(\nu) dS_n(K, \nu)$$

for every $K \in \mathcal{K}^{n+1}$. The function η is uniquely determined up to adding the restriction to $\mathbb{S}^n$ of a linear function.

n-homogeneous valuations on $\mathrm{Conv}_{sc}(\mathbb{R}^n)$

Theorem [Colesanti, Ludwig and Mussnig, 2020]

A functional $Z : \mathrm{Conv}_{sc}(\mathbb{R}^n) \to \mathbb{R}$ is a continuous and epi-translation invariant valuation that is epi-homogeneous of degree n, if and only if there exists $\zeta \in C_0(\mathbb{R}^n)$ such that

$$Z(u) = \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))dx$$

for every $u \in Conv_{sc}(\mathbb{R}^n)$.

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This result can be proved as a consequence
of McMullen's Theorem [Knoerr and U., 2022+]

Are there inequalities for these functionals?

First of all, one needs to work on the family

$$\mathrm{Conv}_0(\mathbb{R}^n) := \{u \in \mathrm{Conv}_{sc}(\mathbb{R}^n) : \partial dom(u) = \{u = 0\}\}.$$

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- Isoperimetric inequalities: if and only if $\frac{\zeta(x)}{\sqrt{1+|x|^2}}$ is bounded away from 0 [Mussnig and U., 2022+].

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In both cases we lose the continuity for the corresponding valuations.

The inequality

For $u \in \mathrm{Conv}_0(\mathbb{R}^n)$ we define

$$V_{n,\zeta}(u) := \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))dx, \quad V_{n+1}(u) := \int_{\mathrm{dom}(u)} |u(x)|dx.$$

Theorem (Mussnig and U., 2022+)

If $\zeta \in C(\mathbb{R}^n), \zeta(x) \geq c\sqrt{1+|x|^2}, c > 0$, then

$$V_{n,\zeta}(u)^{\frac{1}{n}} \geq C(n,\zeta) V_{n+1}(u)^{\frac{1}{n+1}}$$

for every $u \in \mathrm{Conv}_0(\mathbb{R}^n)$.

Hint of proof: Many changes of variables and Wulff's inequality.

THANKS FOR YOUR ATTENTION!

Potential Theory with Multivariate Kernels

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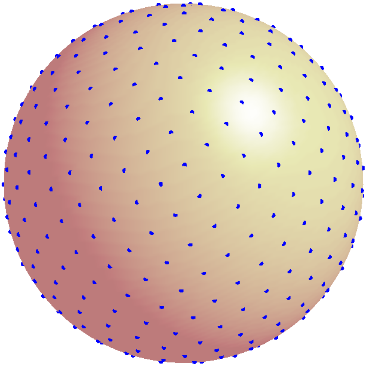
History

In 1904, physicist and Nobel Prize winner J. Thomson worked on a model of the atom – this led to the question: which configuration of electrons on a spherical shell would minimize electrostatic potential energy. Known configurations for $N \in \{1, 2, 3, 4, 5, 6, 12\}$.



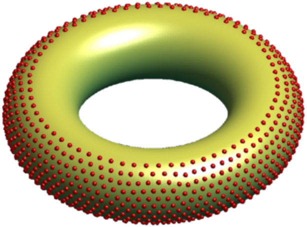
Coulomb Potential: Given a point set $\omega_N := \{x_1, \dots, x_N\}$ on the sphere, minimize

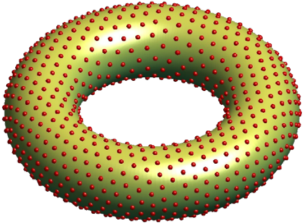
$$\sum_{j \neq s} \frac{1}{\|x_j - x_s\|}.$$

Riesz potential

Let $K : \Omega \times \Omega \to \mathbb{R} \cup \{\infty\}$ where $\Omega = \mathbb{T}^2$, $K(x, y) = f\left(\|x - y\|\right)$ and

$$f(r) = r^{-\alpha}.$$





$\star$ Borodachov, Hardin and Saff: "*Discrete Energy on Rectifiable Sets*" (2019).

Generalization

Let $\omega_N := \{x_1, \dots, x_N\} \subset (\Omega, d)$, with Ω compact and infinite, and $K : \Omega \times \Omega \to \mathbb{R} \cup \{\infty\}$, investigate

$$E_K[\omega_N] = \sum_{j \neq s} K(x_j, x_s).$$

Lemma. Let $N > 1$, then for arbitrary K

$$\frac{\inf_{\omega_N} E_K[\omega_N]}{N(N-1)} \nearrow C \in \mathbb{R} \cup \{\infty\}.$$

Proof. For fixed $x_j \in \omega_{N+1}$

$$E_K[\omega_{N+1}] = E_K[\omega_{N+1} \setminus \{x_j\}] + \sum_{s=1, s\neq j}^{N+1} K(x_j, x_s) + K(x_s, x_j),$$

and summing up over j

$$(N+1) E_K[\omega_{N+1}] \;\geq\; (N+1) \inf_{\omega_N} E_K[\omega_N] + 2 E_K[\omega_{N+1}]. \quad \square$$

Example: Green kernel

Let $(\Omega, d) = (M, g)$ a closed Riemannian manifold, and $\mathcal{G}$ the normalized Green function for the Laplace-Beltrami operator; set

$$K(x,y) = \mathcal{G}(x,y).$$

Theorem. For $M = \mathrm{SO}(3)$, we have

$$-3\pi^{1/3} N^{4/3} \ \leq \ \inf_{\omega_N \subset \mathrm{SO}(3)} E_{\mathcal{G}}(\omega_N) + O(N) \leq \ -4\left(\frac{3}{4}\right)^{4/3} N^{4/3}.$$

$\star$ Beltrán & DF: "*Approximation to uniform distribution in SO(3)*", Constr Approx 52 (2020).

Uniform distribution

Theorem. For a compact Riemannian manifold (M, g) with $\dim(M) > 1$, let G be its normalized Green function, then

$$I_G(\lambda) = \inf_{\mu \in \mathbb{P}(M)} I_G(\mu) = \inf_{\mu \in \mathbb{P}(M)} \iint_M G(x, y) \mathrm{d}\mu(x) \mathrm{d}\mu(y),$$

where λ is the uniform measure on M. Minimizing point sets ω_N for the Green energy satisfy

$$\omega_N \stackrel{w*}{\to} \lambda.$$

$\star$ Beltrán, Corral, Criado Del Rey: "*Discrete and continuous Green energy on compact manifolds*" Journal of Approximation Theory (2019).

Generalization II

A kernel $K : \Omega^2 \to \mathbb{R}$ is called *positive definite* if for every finite signed Borel measure $\mu \in \mathcal{M}(\Omega)$

$$I_K(\mu) = \iint_\Omega K(x,y)\mathrm{d}\mu(x,y) \;\geq\; 0.$$

It is called *conditionally positive definite* if

$$I_K(\mu) \;\geq\; 0$$

for all $\mu \in \mathcal{M}(\Omega)$ with

$$\mu(\Omega) = 0.$$

(One assumes the integrals to make sense.) Sum, limit, and product of PD kernels is again PD.

Convexity of I_K

Lemma. (BHS p.135) Let K be symmetric, lower semi-continuous, and conditionally positive definite. Given $\mu, \nu \in \mathbb{P}(\Omega)$ with

$$I_K(\mu), I_K(\nu) \ < \ \infty,$$

then

$$2I_K(\mu, \nu) \ \leq \ I_K(\mu) + I_K(\nu);$$

where

$$I_K(\mu, \nu) = \iint K(x, y) \mathrm{d}\mu(x) \mathrm{d}\nu(y).$$

Corollary.

$$I_K\big(t\mu + (1-t)\nu\big) \leq tI_K(\mu) + (1-t)I_K(\nu).$$

$\star$ Bilyk, Matzke, Vlasiuk: "*Positive definiteness and the Stolarsky invariance principle.*" arXiv (2021).

Multivariate kernels in Applications

Axilrod-Teller Potential. Let the angle between vectors x, y be denoted by $a(x, y)$

$$K(x, y, z) = \frac{1 + 3a(x, y)a(y, z)a(x, z)}{d(x, y)^3 d(y, z)^3 d(x, z)^3}.$$

$\star$ Axilrod, Teller: "*Interaction of the van der Waals Type Between Three Atoms*", Journal of Chemical Physics. 11 (1943).

Menger Curvature. Let $A(x, y, z)$ be the area of the triangle, spanned by x, y, z.

$$c(x, y, z) = \frac{4\ A(x, y, z)}{d(x, y)d(y, z)d(x, z)}.$$

Stillinger-Weber Potential.
$\star$ Stillinger, Weber: "*Computer simulation of local order in condensed phases of silicon*", Physical Review B. 31 (1985).

Investigated and used for

Kissing Numbers.

⋆ Bachoc, Vallentin: "*New Upper Bounds for Kissing Numbers from Semidefinite Programming*", Journal of the American Mathematical Society 21 (3) (2008).

Energy Minimization.

⋆ Cohn, Woo: "*Three-Point Bounds for Energy Minimization*", Journal of the AMS (25) 4 (2012).

⋆ Bilyk, DF, Glazyrin, Matzke, Park, Vlasiuk: "*Potential theory with multivariate kernels*", Math Z (2022).

Generalization III

A real-valued, symmetric, and continuous kernel K will be called (conditionally) 3-positive definite, if for any fixed $z \in \Omega$, it holds for

$$G_z(x, y) := K(x, y, z).$$

Sum, limit, and product of PD kernels is again PD.

Corollary. $H(x, y) = \int K(x, y, z) \mathrm{d}\mu(z)$ is (conditionally) positive definite, if K is.

Lemma. Let $2 \leq m \leq n-1$, and suppose $H : \Omega^m \to \mathbb{R}$ is continuous, symmetric, and conditionally m-positive definite. Then

$$K(z_1, ..., z_n) := \sum_{1 \leq j_1 < j_2 < \cdots < j_m \leq n} H(z_{j_1}, z_{j_2}, ..., z_{j_m})$$

is conditionally n-positive definite.

Some results

Lemma. Suppose K is symmetric, continuous, and (conditionally) PD, then for $\mu_j \in \mathbb{P}(\Omega)$

$$I_K(\mu_1, \ldots, \mu_n) \leq \frac{1}{n} \sum_{j=1}^{n} I_K(\mu_j).$$

Corollary. I_K is convex.

Now let $\Omega = \mathbb{S}^2$, and K be rotationally invariant, i.e. have the form

$$K(x_1, \ldots, x_n) = F\Big((\langle x_i, x_j \rangle)_{i,j=1}^{n} \Big).$$

Some results II

Theorem. Suppose that $K : (\mathbb{S}^2)^n \to \mathbb{R}$ is continuous, symmetric, rotationally invariant, and conditionally n-positive definite on $\mathbb{S}^2$. Then σ is a minimizer of I_K over $\mathbb{P}(\mathbb{S}^2)$.

We will write $K(x, y, z) = F(u, v, t)$ where

$$u = \langle x, y \rangle, \quad v = \langle z, y \rangle, \quad t = \langle x, z \rangle.$$

Corollary. Let $f : [-1, 1] \to \mathbb{R}$ be a real-analytic function with nonnegative Maclaurin coefficients and let $F(u, v, t) = f(uvt)$. Then the uniform surface measure σ minimizes the energy I_K over $\mathbb{P}(\mathbb{S}^2)$.

Thank you for your Time

Minimizing p-Frame Energies and Mixed Volumes

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The research in this presentation is in collaboration with Dmitriy Bilyk, Alexey Glazyrin, Josiah Park, and Oleksandr Vlasiuk.

Energy on the Sphere

Let $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$. Given a continuous (potential) function $F : [-1, 1] \to \mathbb{R}$, the **(discrete) energy** of a configuration (multiset) $\omega_N = \{z_1, \ldots, z_N\} \subset \mathbb{S}^{d-1}$ is

$$E_F(\omega_N) = \frac{1}{N^2} \sum_{i,j=1}^{N} F(\langle z_i, z_j \rangle),$$

and the **(continuous) energy** of a probability measure $\mu \in \mathbb{P}(\mathbb{S}^{d-1})$ is

$$I_F(\mu) = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} F(\langle x, y \rangle) d\mu(x) d\mu(y).$$

- What is the mimimal energy (for fixed N for E_F)?
- Is the uniform measure σ a minimizer of I_F? Is the support of any minimizer of a lower dimension? Discrete?
- Are minimizers of E_F uniformly distributed? Well-separated? Do they concentrate and form “clumps"? What happens as $N \to \infty$?

Energy on the Sphere

Let $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$. Given a continuous (potential) function $F : [-1, 1] \to \mathbb{R}$, the **(discrete) energy** of a configuration (multiset) $\omega_N = \{z_1, \ldots, z_N\} \subset \mathbb{S}^{d-1}$ is

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and the **(continuous) energy** of a probability measure $\mu \in \mathbb{P}(\mathbb{S}^{d-1})$ is

$$I_F(\mu) = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} F(\langle x, y \rangle) d\mu(x) d\mu(y).$$

- If $\mu_{\omega_N} = \frac{1}{N} \sum_{j=1}^{N} \delta_{z_j}$, then

$$I_F(\mu_{\omega_N}) = \frac{1}{N^2} \sum_{i,j=1}^{N} F(\langle z_i, z_j \rangle) = E_F(\omega_N).$$

- The weak* density of the linear span of Dirac masses in $\mathbb{P}(\mathbb{S}^{d-1})$ gives

$$\lim_{N \to \infty} \min_{\omega_N \subset \mathbb{S}^{d-1}} E_F(\omega_N) = \inf_{\mu \in \mathbb{P}(\mathbb{S}^{d-1})} I_F(\mu).$$

Riesz s-energies

For $s \in \mathbb{R}$, we define the Riesz kernel as

$$R_s(\langle x, y\rangle) = \begin{cases} \frac{1}{\|x-y\|^s} & s > 0 \\ -\log(\|x-y\|) & s = 0 \\ -\|x-y\|^{-s} & s < 0 \end{cases}$$

Coulomb ($s = d - 2$), Logarithmic ($s = 0$), Euclidean distance ($s = -1$).

Theorem (Björck, 1956)

The minimizers of I_{R_s} are

- σ *(uniquely) if* $-2 < s < d$
- *Any measure with center of mass at the origin if* $s = -2$
- *Any measure of the form* $\frac{1}{2}(\delta_p + \delta_{-p})$ *if* $s < -2$.

Theorem (Classical; Götz, Hardin, Kuijlaars, Saff)

If $s > -2$*, the minimizers of* E_{R_s} *are uniformly distributed on the sphere.*

p-Frame Energy

Stronger repulsion tends to lead to minimizers "spreading out" while weaker repulsion leads to the support concentrating.

Theorem (Carillo, Figalli, Patacchini, 2017)

Suppose $F(\langle x, y\rangle) = G(\|x-y\|)$ and $G'(t) \sim -t^{\alpha-1}$ as $t \to 0$ for some $\alpha > 2$. If μ is a minimizer of I_F, then μ has discrete (finite) support.

For $p \in (0, \infty)$, we define the p-**frame potential** as

$$F_p(\langle x, y\rangle) = |\langle x, y\rangle|^p.$$

Minimizing this energy for $p = 2$ results in tight frames/isotropic measures and for $p = 4$ (in the complex setting) results in symmetric information complete positive operator-valued measures (SIC-POVM's).

Since $|\langle x, y\rangle|^p = 1 - \frac{p}{2}\|x-y\|^2 + O(\|x-y\|^4)$, it falls into the limit case $\alpha = 2$. We might expect the types of minimizers to vary with p.

p-Frame Energy

Theorem (Bilyk, Glazyrin, Matzke, Park, Vlasiuk, 2021)

If $p \in 2\mathbb{N}$, σ is a minimizer of I_{F_p}. If $p \notin 2\mathbb{N}$ and μ is a minimizer, then $(\mathrm{supp}(\mu))^\circ = \emptyset$.

Conjecture (Bilyk, Glazyrin, Matzke, Park, Vlasiuk)

If $p \notin 2\mathbb{N}$, then the minimizers of the p-frame energy are discrete.

Theorem (Bilyk, Glazyrin, Matzke, Park, Vlasiuk, 2022)

If C is a tight $(2m+1)$-design on $\mathbb{S}^{d-1}$ and $p \in (2m-2, 2m)$, then $\mu = \frac{1}{\#C} \sum_{x \in C} \delta_x$ is a minimizer of I_{F_p}. Moreover, when this happens, all minimizers of I_{F_p} are discrete.

Tight Designs

A **spherical k-design** is a set of points $\{x_1, ..., x_N\} \subset \mathbb{S}^{d-1}$ such that

$$\int_{\mathbb{S}^{d-1}} q(x)d\sigma(x) = \frac{1}{N}\sum_{i=1}^{N} q(x_i)$$

for all polynomials q on $\mathbb{R}^d$ of degree at most k. A spherical $(2m+1)$-design is **tight** if it is centrally symmetric and there are $m+2$ inner products between its points.

d	$\lvert C\rvert$	p-range	Configuration
d	$2d$	$(0,2)$	cross polytope
2	$2k$	$(2k-4, 2k-2)$	$2k$-gon
3	12	$(2,4)$	icosahedron
7	56	$(2,4)$	kissing configuration
8	240	$(4,6)$	E_8 roots
23	552	$(2,4)$	equiangular lines
23	4600	$(4,6)$	kissing configuration
24	196560	$(8,10)$	Leech lattice

L^p-mixed Volumes

Let $C \subset \mathbb{R}^d$ be a convex body,

$$\sigma_C(B) = |\{x \in \partial C : n_x \in B\}|_{d-1}$$

for all Borel $B \subseteq \mathbb{S}^{d-1}$, and h_C be the **support function** of C

$$h_C(y) = \sup_{x \in C} \langle x, y \rangle.$$

Given two convex bodies, C and D, and $p \geq 1$, we define the L^p**-mixed volume** of the two to be

$$V_p(C, D) = \frac{p}{d} \lim_{\varepsilon \to 0} \frac{|C +_p \varepsilon D|_d - |C|_d}{\varepsilon} = \frac{1}{d} \int_{\mathbb{S}^{d-1}} h_D(x)^p h_C(x)^{1-p} d\sigma_C(x),$$

where $C +_p \varepsilon D$ is the convex body with support function

$$h_{C +_p \varepsilon D}(x) = \sqrt[p]{h_C(x)^p + \varepsilon h_D(x)^p}.$$

Mixed Volumes with Projection Bodies

The L^p**-projection body** of C, $\Pi_p C$, is the origin-symmetric convex body with support function

$$h_{\Pi_p C}(x) = \left(c_{d,p} \int_{\mathbb{S}^{d-1}} |\langle x, y \rangle|^p h_C(x)^{1-p} d\sigma_C(x) \right)^{\frac{1}{p}}.$$

Defining $\sigma_{C,p}$ such that $d\sigma_{C,p}(x) = h_C(x)^{1-p} d\sigma_C(x)$, we see that

$$\begin{aligned} I_{F_p}(\sigma_{C,p}) &= \iint_{\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}} |\langle x, y \rangle|^p d\sigma_{C,p}(x) d\sigma_{C,p}(y) \\ &= \frac{1}{c_{d,p}} \int_{\mathbb{S}^{d-1}} h_{\Pi_p C}(x)^p h_C(x)^{1-p} d\sigma_C(x) = \frac{d}{c_{d,p}} V_p(C, \Pi_p C). \end{aligned}$$

Thus, minimizing the p-frame energy (over admissible measures) is the same as minimizing $V_p(C, \Pi_p C)$ over all symmetric convex bodies C (scaled to satisfy $\sigma_{C,p}(\mathbb{S}^{d-1}) = 1$).

Proposition (Bilyk, Glazyrin, Matzke, Park, Vlasiuk, 2022)

The quantity $\frac{V_1(C, \Pi_1 C)}{|\partial C|_{d-1}^2}$ is minimized if and only if C is a hypercube.

Thank you!

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