About Bezout inequalities for mixed volumes

M. Szusterman,

Université de Paris

May 2022

Workshop in Convexity and higher-dimensional probability,
Atlanta
Mixed volume : Minkowski’s definition

Denote by $\mathcal{K}_n = \{ K \subset \mathbb{R}^n : K \text{ compact convex set} \}$.

Let $K, L \in \mathcal{K}_n$. Then $\text{Vol}_n(\lambda K + \mu L)$ is a polynomial in $(\lambda, \mu)$:

$$\text{Vol}_n(\lambda K + \mu L) = \sum_{k=0}^{n} \binom{n}{k} v_k \lambda^k \mu^{n-k}$$

where $v_k = V_n(K[k], L[n-k]) = V_n(K, \ldots, K, L, \ldots, L)$ are called mixed volumes.
Mixed volume: Minkowski’s definition

Let $K, L \in \mathcal{K}_n$. Then $Vol_n(\lambda K + \mu L) = \sum_{k=0}^{n} \binom{n}{k} \nu_k \lambda^k \mu^{n-k}$

Let $K_1, \ldots, K_m \in \mathcal{K}_n$. Then:

$$Vol_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{a = (a_1, \ldots, a_m)}^{\text{where } |a| = n} \binom{n}{a} \nu_a \lambda^a$$

where $\nu_a = V_n(K_1[a_1], \ldots, K_m[a_m])$ are called mixed volumes.
Mixed volume: Minkowski’s definition

Let $K, L \in \mathcal{K}_n$. Then $Vol_n(\lambda K + \mu L) = \sum_{k=0}^{n} \binom{n}{k} v_k \lambda^k \mu^{n-k}$

Let $K_1, \ldots, K_m \in \mathcal{K}_n$. Then:

$$Vol_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{a=(a_1, \ldots, a_m), |a|=n} \binom{n}{a} v_a \lambda^a$$

where $v_a = V_n(K_1[a_1], \ldots, K_m[a_m])$ are called mixed volumes.

$V_n : \mathcal{K}_n^n \to [0, +\infty)$ is a multilinear, continuous functional.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transform. Then:

$$V_n(TK_1, \ldots, TK_n) = det(T) V_n(K_1, \ldots, K_n)$$
Bezout inequality

Let $f_1, \ldots, f_r : \mathbb{R}^n \to \mathbb{R}$ be polynomials. Denote by $X_1, \ldots, X_r$ the associated algebraic varieties $(X_i := \{x \in \mathbb{R}^n : f_i(x) = 0\})$.

The Bezout inequality states that:

$$\deg(X_1 \cap \cdots \cap X_r) \leq \prod \deg(X_i) \quad [B]$$
Bezout inequality

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The \textit{Bezout inequality} states that:

\[
\deg(X_1 \cap \ldots \cap X_r) \leq \prod \deg(X_i) \quad \text{[B]}
\]

Denote by \( P_1, \ldots, P_r \) the Newton polytopes of \( f_1, \ldots, f_r \)
Bezout inequality

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The *Bezout inequality* states that:

$$\deg(X_1 \cap \ldots \cap X_r) \leq \prod \deg(X_i) \quad [B]$$

Denote by $P_1, \ldots, P_r$ the Newton polytopes of $f_1, \ldots, f_r$.

We can reformulate [B] within the language of mixed volumes:

$$V(P_1, \ldots, P_r, \Delta[n-r])V(\Delta)^{r-1} \leq \prod_{i=1}^{r} V(P_i, \Delta[n-1])$$

thanks to a theorem by Bernstein, Kushnirenko and Khovanskii.
Let $f_1, \ldots, f_n : \mathbb{R}^n \to \mathbb{R}$ be polynomials.
Let $X = X_2 \cap \ldots \cap X_n$ of dimension 1, and $Y = X_1$ (codim.1).
Then Bezout inequality:

$$\text{deg}(X \cap Y) \leq \text{deg}(X) \text{deg}(Y) \quad [B]$$

translates to

$$V_n(P_1, \ldots, P_n) V_n(\Delta) \leq V_n(P_2, \ldots, P_n, \Delta) V_n(P_1, \Delta[n - 1]).$$

(which allows to recover previous inequality [B])
Relaxed Bezout inequality

- for the $n$-simplex $\Delta$:

$$V(L_1, ..., L_n) V(\Delta) \leq V(L_2, ..., L_n, \Delta) V(L_1, \Delta[n - 1]).$$

- Thanks to Diskant inequality, J. Xiao has shown (2019):

$$V(L_1, ..., L_n) V(K) \leq n V(L_2, ..., L_n, K) V(L_1, K[n - 1])$$

for any convex bodies $L_1, ..., L_n$, and for any $K$. 
Bezout constants

We define:

\[ b_2(K) = \max_{L_1, L_2} \frac{V(L_1, L_2, K[n-2])V(K)}{V(L_1, K[n-1])V(L_2, K[n-1])} \geq 1 \]

And similarly

\[ b(K) = \max_{L_1, \ldots, L_n} \frac{V(L_1, \ldots, L_n)V(K)}{V(L_2, \ldots, L_n, K)V(L_1, K[n-1])} \geq 1 \]

So that:

- \( b_2(\Delta) = b(\Delta) = 1 \);
- \( \forall K, 1 \leq b_2(K) \leq b(K) \);
- by [Diskant, Xiao] : \( \max_K b(K) \leq n \).
- \( \forall K, b(TK) = b(K) \), for any (full-rank) affine \( T \).
Who are the minimizers?

**Question [SZ ’15]**

For which bodies do we have $b_2(K) = 1$?

**Question [SSZ ’18]**

For which bodies do we have $b(K) = 1$?

**SZ ’15 →** [Soprunov, Zvavitch] (2015)

**SSZ ’18 →** [Saroglou, Soprunov, Zvavitch] (2018)
Who are the minimizers?

**Qstn [SZ '15]** For which $K$, do we have $b_2(K) = 1$?

**Theorem [SSZ '18]** If $b(K) = 1$, then $K = \Delta$.

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Qstn [SSZ ’18] For which $K$ do we have $b(K) = 1$?

- **Theorem** [SSZ ’18] If $b(K) = 1$, then $K = \Delta$.
- this doesn’t close former question, since $b_2(K) \leq b(K)$. 
Who are the minimizers?

Qstn [SZ '15] For which $K$, do we have $b_2(K) = 1$?

- **Theorem [SSZ '18]** If $b_2(P) = 1$, then $P = \Delta$.
  (where $P$ is an $n$-polytope)

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► Theorem [SSZ ’18] If $b_2(P) = 1$, then $P = \Delta$. (where $P$ is an $n$-polytope)

► Prop [SZ ’15] if $b_2(K) = 1$, then $K \neq A + B$ (with $A \neq B$) ($K$ cannot be decomposable)

Qstn [SSZ ’18] For which $K$ do we have $b(K) = 1$?

► Theorem [SSZ ’18] If $b(K) = 1$, then $K = \Delta$. 
Who are the minimizers?

**Qstn** [SZ ’15] For which $K$, do we have $b_2(K) = 1$?

- **Thm** [SSZ ’18] Let $P \in \text{Poly}_n$. Then $b_2(P) = 1 \Rightarrow P = \Delta$.
- **Thm** [’15, ’18] if $b_2(K) = 1$, then $K$ cannot be weakly decomposable ($\Rightarrow K \notin \mathcal{W}_n$).

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  $\rightarrow$ excludes bodies with (somewhere) smooth boundary.

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\[ \Rightarrow \] recovers characterization among polytopes, since $\text{Poly}_n \cap \mathcal{W}_n = \text{Poly}_n \setminus \{\Delta\}$.

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- **Thm** [SSZ ’18] Let $P \in \textbf{Poly}_n$. Then $b_2(P) = 1 \Rightarrow P = \Delta$.
- **Thm** [’15, ’18] if $b_2(K) = 1$, then $K$ cannot be *weakly decomposable* ($\rightarrow K \notin \mathcal{W}_n$)

- … some more *restrictions*, eg: at most *finitely* many facets.

**Qstn** [SSZ ’18] For which $K$ do we have $b(K) = 1$?

- **Theorem** [SSZ ’18] If $b(K) = 1$, then $K = \Delta$.
  
  $\rightarrow$ proof uses *Wulff shape* bodies, a pointwise Aleksandrov differentiation lemma, and builds on above *restrictions*. 
A new necessary condition

Let \( L \in \mathcal{K}_n \) be a \( k \)-dimensional. Denote:

\[
Iso(L) := \frac{1}{k} \frac{Vol_{k-1}(\partial L)}{Vol_k(L)} =: \frac{1}{k} \frac{\mid \partial L \mid}{\mid L \mid}
\]

**Thm [S. 2022]** If \( b_2(K) = 1 \), then:

For any facet \( F \) of \( K \): \( Iso(F) \leq Iso(K) \).

\( \rightarrow \) recovers the “at most finitely many facets” restriction.
A new necessary condition

Let $L \in \mathcal{K}_n$ be a $k$-dimensional. Denote:

$$\text{Iso}(L) := \frac{1}{k} \frac{\text{Vol}_{k-1}(\partial L)}{\text{Vol}_k(L)} =: \frac{1}{k} \frac{|\partial L|}{|L|}$$

**Thm** [S. 2022] If $b_2(K) = 1$, then, for any affine transform $T$:

For any facet $F$ of $K$:

$$\text{Iso}(TF) \leq \text{Iso}(TK).$$

(since $b_2(K)$ is affine invariant, while $\max_F \frac{\text{Iso}(F)}{\text{Iso}(K)}$, is not)
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**Thm**[S. 2022] If $b_2(K) = 1$, then, for any affine transform $T$:

For any facet $F$ of $K$: $Iso(TF) \leq Iso(TK)$.

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**Question**: if $P \neq \Delta$, does there always exist an affine transform $T$ s.t. $\max_F \frac{Iso(TF)}{Iso(TP)} > 1$?
... any questions?

Thank you for your attention!!
\( \mathcal{K}^{n+1} := \{ \text{compact convex bodies in } \mathbb{R}^{n+1} \} \)

with the topology induced by the Hausdorff distance.
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with the topology induced by the Hausdorff distance.

\[ \text{Conv}_{sc}(\mathbb{R}^n) := \{ u : \mathbb{R}^n \to \mathbb{R} \cup \{ +\infty \} : \text{convex, l.s.c. and proper, } \lim_{|x| \to \infty} \frac{u(x)}{|x|} = +\infty \} \]

with the topology induced by \textit{epi}-convergence:

\[ u_j \xrightarrow{\text{epi}} u \text{ if} \]

- For every sequence \((x_j)\) that converges to \(x\), \(u(x) \leq \lim \inf_{j \to \infty} u_j(x_j)\).
- There exists a sequence \((x_j)\) converging to \(x\) such that \(u(x) = \lim_{j \to \infty} u_j(x_j)\).
...and their Valuations

Valuations on $\mathcal{K}^{n+1}$:

Functionals $Y : \mathcal{K}^{n+1} \to \mathbb{R}$ such that for every $K, L \in \mathcal{K}^{n+1}$, $K \cup L \in \mathcal{K}^{n+1}$

$$Y(K \cup L) + Y(K \cap L) = Y(K) + Y(L).$$
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Valuations on $\text{Conv}_{sc}(\mathbb{R}^n)$:
Functionals $Z : \text{Conv}_{sc}(\mathbb{R}^n) \to \mathbb{R}$ such that for every $u, v \in \text{Conv}_{sc}(\mathbb{R}^n)$, $u \land v \in \text{Conv}_{sc}(\mathbb{R}^n)$

$$Z(u \land v) + Z(u \lor v) = Z(u) + Z(v).$$

[Ludwig, Alesker, Colesanti, Mussnig, Knoerr...]

...and their Valuations
**Theorem [McMullen, 1980]**

A functional $Y : \mathcal{K}^{n+1} \to \mathbb{R}$ is a continuous, translation invariant real valued valuation which is $n$-homogeneous, if and only if there exists a continuous function $\eta : \mathbb{S}^n \to \mathbb{R}$ such that

$$Y(K) = \int_{\mathbb{S}^n} \eta(\nu) dS_n(K, \nu)$$

for every $K \in \mathcal{K}^{n+1}$. The function $\eta$ is uniquely determined up to adding the restriction to $\mathbb{S}^n$ of a linear function.
A functional $Z : \text{Conv}_{sc}(\mathbb{R}^n) \to \mathbb{R}$ is a continuous and epi-translation invariant valuation that is epi-homogeneous of degree $n$, if and only if there exists $\zeta \in C_0(\mathbb{R}^n)$ such that

$$Z(u) = \int_{\text{dom}(u)} \zeta(\nabla u(x)) \, dx$$

for every $u \in \text{Conv}_{sc}(\mathbb{R}^n)$. 

This result can be proved as a consequence of McMullen's Theorem [Knoerr and U., 2022+].
n-homogeneous valuations on Conv_{sc}(\mathbb{R}^n)

**Theorem** [Colesanti, Ludwig and Mussnig, 2020]

A functional \( Z : \text{Conv}_{sc}(\mathbb{R}^n) \to \mathbb{R} \) is a continuous and epi-translation invariant valuation that is epi-homogeneous of degree \( n \), if and only if there exists \( \zeta \in C_0(\mathbb{R}^n) \) such that

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This result can be proved as a consequence of McMullen’s Theorem [Knoerr and U., 2022+]
Are there inequalities for these functionals?

First of all, one needs to work on the family

\[ \text{Conv}_0(\mathbb{R}^n) := \{ u \in \text{Conv}_{sc}(\mathbb{R}^n) : \partial \text{dom}(u) = \{ u = 0 \} \}. \]
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- Brunn-Minkowski type inequalities: if and only if \( \zeta \) is a real valued convex function. Consequence of [Colesanti, Hug and Saorin-Gomez, 2014]. Already studied by [Klartag, 2005].
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- Isoperimetric inequalities: if and only if $\frac{\zeta(x)}{\sqrt{1+|x|^2}}$ is bounded away from 0 [Mussnig and U., 2022+].
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In both cases we lose the continuity for the corresponding valuations.
The inequality

For $u \in \text{Conv}_0(\mathbb{R}^n)$ we define

$$V_{n,\zeta}(u) := \int_{\text{dom}(u)} \zeta(\nabla u(x)) \, dx, \quad V_{n+1}(u) := \int_{\text{dom}(u)} |u(x)| \, dx.$$ 

**Theorem (Mussnig and U., 2022+)**

If $\zeta \in C(\mathbb{R}^n)$, $\zeta(x) \geq c \sqrt{1 + |x|^2}$, $c > 0$, then

$$V_{n,\zeta}(u)^{\frac{1}{n}} \geq C(n, \zeta) V_{n+1}(u)^{\frac{1}{n+1}}$$

for every $u \in \text{Conv}_0(\mathbb{R}^n)$.

**Hint of proof:** Many changes of variables and Wulff’s inequality.
THANKS FOR YOUR ATTENTION!
Potential Theory with Multivariate Kernels

Damir Ferizović

Department of Mathematics
KU Leuven
History

In 1904, physicist and Nobel Prize winner J. Thomson worked on a model of the atom – this led to the question: which configuration of electrons on a spherical shell would minimize electrostatic potential energy. Known configurations for \( N \in \{1, 2, 3, 4, 5, 6, 12\} \).

**Coulomb Potential:** Given a point set \( \omega_N := \{x_1, \ldots, x_N\} \) on the sphere, minimize

\[
\sum_{j \neq s} \frac{1}{\|x_j - x_s\|}.
\]
Riesz potential

Let $K : \Omega \times \Omega \to \mathbb{R} \cup \{\infty\}$ where $\Omega = \mathbb{T}^2$, $K(x, y) = f(\|x - y\|)$ and

$$f(r) = r^{-\alpha}.$$ 

Generalization

Let $\omega_N := \{x_1, \ldots, x_N\} \subset (\Omega, d)$, with $\Omega$ compact and infinite, and $K : \Omega \times \Omega \to \mathbb{R} \cup \{\infty\}$, investigate

$$E_K[\omega_N] = \sum_{j \neq s} K(x_j, x_s).$$

**Lemma.** Let $N > 1$, then for arbitrary $K$

$$\inf_{\omega_N} \frac{E_K[\omega_N]}{N(N - 1)} \uparrow C \in \mathbb{R} \cup \{\infty\}.$$

**Proof.** For fixed $x_j \in \omega_{N+1}$

$$E_K[\omega_{N+1}] = E_K[\omega_{N+1} \setminus \{x_j\}] + \sum_{s=1, s \neq j}^{N+1} K(x_j, x_s) + K(x_s, x_j),$$

and summing up over $j$

$$(N + 1)E_K[\omega_{N+1}] \geq (N + 1) \inf_{\omega_N} E_K[\omega_N] + 2E_K[\omega_{N+1}]. \qed$$
Example: Green kernel

Let $(\Omega, d) = (M, g)$ a closed Riemannian manifold, and $\mathcal{G}$ the normalized Green function for the Laplace-Beltrami operator; set

$$K(x, y) = \mathcal{G}(x, y).$$

**Theorem.** For $M = \text{SO}(3)$, we have

$$-3\pi^{1/3}N^{4/3} \leq \inf_{\omega_N \subset \text{SO}(3)} E_G(\omega_N) + O(N) \leq -4 \left(\frac{3}{4}\right)^{4/3} N^{4/3}.$$ 

**Uniform distribution**

**Theorem.** For a compact Riemannian manifold \((M, g)\) with \(\text{dim}(M) > 1\), let \(G\) be its normalized Green function, then

\[
I_G(\lambda) = \inf_{\mu \in \mathcal{P}(M)} I_G(\mu) = \inf_{\mu \in \mathcal{P}(M)} \iint_M G(x, y) d\mu(x) d\mu(y),
\]

where \(\lambda\) is the uniform measure on \(M\). Minimizing point sets \(\omega_N\) for the Green energy satisfy

\[
\omega_N \overset{w^*}{\to} \lambda.
\]

A kernel $K : \Omega^2 \to \mathbb{R}$ is called \emph{positive definite} if for every finite signed Borel measure $\mu \in \mathcal{M}(\Omega)$

$$I_K(\mu) = \int\int_{\Omega} K(x, y) \, d\mu(x, y) \geq 0.$$ 

It is called \emph{conditionally positive definite} if

$$I_K(\mu) \geq 0$$

for all $\mu \in \mathcal{M}(\Omega)$ with

$$\mu(\Omega) = 0.$$ 

(One assumes the integrals to make sense.) Sum, limit, and product of PD kernels is again PD.
Convexity of $l_K$

**Lemma.** (BHS p.135) Let $K$ be symmetric, lower semi-continuous, and conditionally positive definite. Given $\mu, \nu \in \mathbb{P}(\Omega)$ with

$$l_K(\mu), l_K(\nu) < \infty,$$

then

$$2l_K(\mu, \nu) \leq l_K(\mu) + l_K(\nu);$$

where

$$l_K(\mu, \nu) = \int\int K(x, y)d\mu(x)d\nu(y).$$

**Corollary.**

$$l_K(t \mu + (1 - t)\nu) \leq tl_K(\mu) + (1 - t)l_K(\nu).$$

Multivariate kernels in Applications

Axilrod-Teller Potential. Let the angle between vectors $x, y$ be denoted by $a(x, y)$

$$K(x, y, z) = \frac{1 + 3a(x, y)a(y, z)a(x, z)}{d(x, y)^3d(y, z)^3d(x, z)^3}.$$  

Axilrod, Teller: "Interaction of the van der Waals Type Between Three Atoms", Journal of Chemical Physics. 11 (1943).

Menger Curvature. Let $A(x, y, z)$ be the area of the triangle, spanned by $x, y, z$.

$$c(x, y, z) = \frac{4A(x, y, z)}{d(x, y)d(y, z)d(x, z)}.$$

Stillinger-Weber Potential.

Kissing Numbers.

Energy Minimization.

A real-valued, symmetric, and continuous kernel $K$ will be called (conditionally) $3$-positive definite, if for any fixed $z \in \Omega$, it holds for

$$G_z(x, y) := K(x, y, z).$$

Sum, limit, and product of PD kernels is again PD.

**Corollary.** $H(x, y) = \int K(x, y, z) \, d\mu(z)$ is (conditionally) positive definite, if $K$ is.

**Lemma.** Let $2 \leq m \leq n - 1$, and suppose $H : \Omega^m \to \mathbb{R}$ is continuous, symmetric, and conditionally $m$-positive definite. Then

$$K(z_1, \ldots, z_n) := \sum_{1 \leq j_1 < j_2 < \cdots < j_m \leq n} H(z_{j_1}, z_{j_2}, \ldots, z_{j_m})$$

is conditionally $n$-positive definite.
Some results

**Lemma.** Suppose $K$ is symmetric, continuous, and (conditionally) PD, then for $\mu_j \in \mathbb{P}(\Omega)$

$$l_K(\mu_1, \ldots, \mu_n) \leq \frac{1}{n} \sum_{j=1}^{n} l_K(\mu_j).$$

**Corollary.** $l_K$ is convex.

Now let $\Omega = S^2$, and $K$ be rotationally invariant, i.e. have the form

$$K(x_1, \ldots, x_n) = F\left( (\langle x_i, x_j \rangle)_{i,j=1}^{n} \right).$$
Some results II

**Theorem.** Suppose that $K : (S^2)^n \to \mathbb{R}$ is continuous, symmetric, rotationally invariant, and conditionally $n$-positive definite on $S^2$. Then $\sigma$ is a minimizer of $I_K$ over $\mathbb{P}(S^2)$.

We will write $K(x, y, z) = F(u, v, t)$ where

$$u = \langle x, y \rangle, \quad v = \langle z, y \rangle, \quad t = \langle x, z \rangle.$$

**Corollary.** Let $f : [-1, 1] \to \mathbb{R}$ be a real-analytic function with nonnegative Maclaurin coefficients and let $F(u, v, t) = f(uvt)$. Then the uniform surface measure $\sigma$ minimizes the energy $I_K$ over $\mathbb{P}(S^2)$. 
Thank you for your Time
Minimizing $p$-Frame Energies and Mixed Volumes

Ryan W. Matzke

Technische Universität Graz

The research in this presentation is in collaboration with Dmitriy Bilyk, Alexey Glazyrin, Josiah Park, and Oleksandr Vlasiuk.
Let $S^{d-1}$ be the unit sphere in $\mathbb{R}^d$. Given a continuous (potential) function $F : [-1, 1] \to \mathbb{R}$, the (discrete) energy of a configuration (multiset) $\omega_N = \{z_1, \ldots, z_N\} \subset S^{d-1}$ is

$$E_F(\omega_N) = \frac{1}{N^2} \sum_{i,j=1}^{N} F(\langle z_i, z_j \rangle),$$

and the (continuous) energy of a probability measure $\mu \in \mathbb{P}(S^{d-1})$ is

$$I_F(\mu) = \int_{S^{d-1}} \int_{S^{d-1}} F(\langle x, y \rangle) d\mu(x) d\mu(y).$$

- What is the minimal energy (for fixed $N$ for $E_F$)?
- Is the uniform measure $\sigma$ a minimizer of $I_F$? Is the support of any minimizer of a lower dimension? Discrete?
- Are minimizers of $E_F$ uniformly distributed? Well-separated? Do they concentrate and form “clumps”? What happens as $N \to \infty$?
Energy on the Sphere

Let $S^{d-1}$ be the unit sphere in $\mathbb{R}^d$. Given a continuous (potential) function $F : [-1, 1] \to \mathbb{R}$, the (discrete) energy of a configuration (multiset) $\omega_N = \{z_1, \ldots, z_N\} \subset S^{d-1}$ is

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$$I_F(\mu) = \int_{S^{d-1}} \int_{S^{d-1}} F(\langle x, y \rangle) d\mu(x) d\mu(y).$$

- If $\mu_{\omega_N} = \frac{1}{N} \sum_{j=1}^{N} \delta_{z_j}$, then

$$I_F(\mu_{\omega_N}) = \frac{1}{N^2} \sum_{i,j=1}^{N} F(\langle z_i, z_j \rangle) = E_F(\omega_N).$$

- The weak* density of the linear span of Dirac masses in $\mathbb{P}(S^{d-1})$ gives

$$\lim_{N \to \infty} \min_{\omega_N \subset S^{d-1}} E_F(\omega_N) = \inf_{\mu \in \mathbb{P}(S^{d-1})} I_F(\mu).$$
Riesz $s$-energies

For $s \in \mathbb{R}$, we define the Riesz kernel as

$$R_s(\langle x, y \rangle) = \begin{cases} 
\frac{1}{\|x-y\|^s} & s > 0 \\
-\log(\|x-y\|) & s = 0 \\
-\|x-y\|^{-s} & s < 0
\end{cases}$$

Coulomb ($s = d - 2$), Logarithmic ($s = 0$), Euclidean distance ($s = -1$).

**Theorem (Björck, 1956)**

*The minimizers of $I_{R_s}$ are*

- $\sigma$ (uniquely) if $-2 < s < d$
- Any measure with center of mass at the origin if $s = -2$
- Any measure of the form $\frac{1}{2}(\delta_p + \delta_{-p})$ if $s < -2$.

**Theorem (Classical; Götz, Hardin, Kuijlaars, Saff)**

*If $s > -2$, the minimizers of $E_{R_s}$ are uniformly distributed on the sphere.*
Stronger repulsion tends to lead to minimizers “spreading out” while weaker repulsion leads to the support concentrating.

Theorem (Carillo, Figalli, Patacchini, 2017)

Suppose $F(\langle x, y \rangle) = G(\|x - y\|)$ and $G'(t) \sim -t^{\alpha-1}$ as $t \to 0$ for some $\alpha > 2$. If $\mu$ is a minimizer of $I_F$, then $\mu$ has discrete (finite) support.

For $p \in (0, \infty)$, we define the $p$-frame potential as

$$F_p(\langle x, y \rangle) = |\langle x, y \rangle|^p.$$

Minimizing this energy for $p = 2$ results in tight frames/isotropic measures and for $p = 4$ (in the complex setting) results in symmetric information complete positive operator-valued measures (SIC-POVM’s).

Since $|\langle x, y \rangle|^p = 1 - \frac{p}{2}\|x - y\|^2 + O(\|x - y\|^4)$, it falls into the limit case $\alpha = 2$. We might expect the types of minimizers to vary with $p$.
Theorem (Bilyk, Glazyrin, Matzke, Park, Vlasiuk, 2021)

If \( p \in 2\mathbb{N} \), \( \sigma \) is a minimizer of \( I_{F_p} \). If \( p \not\in 2\mathbb{N} \) and \( \mu \) is a minimizer, then 
\[
(supp(\mu))^\circ = \emptyset.
\]

Conjecture (Bilyk, Glazyrin, Matzke, Park, Vlasiuk)

If \( p \not\in 2\mathbb{N} \), then the minimizers of the \( p \)-frame energy are discrete.

Theorem (Bilyk, Glazyrin, Matzke, Park, Vlasiuk, 2022)

If \( C \) is a tight \((2m + 1)\)-design on \( \mathbb{S}^{d-1} \) and \( p \in (2m - 2, 2m) \), then 
\[
\mu = \frac{1}{\#C} \sum_{x \in C} \delta_x
\]
is a minimizer of \( I_{F_p} \). Moreover, when this happens, all minimizers of \( I_{F_p} \) are discrete.
A spherical $k$-design is a set of points $\{x_1, \ldots, x_N\} \subset S^{d-1}$ such that

$$\int_{S^{d-1}} q(x) d\sigma(x) = \frac{1}{N} \sum_{i=1}^{N} q(x_i)$$

for all polynomials $q$ on $\mathbb{R}^d$ of degree at most $k$. A spherical $(2m + 1)$-design is tight if it is centrally symmetric and there are $m + 2$ inner products between its points.

| $d$  | $|C|$ | $p$-range $|$ Configuration               |
|------|------|----------------------------------------|
| $d$  | $2d$ | $(0, 2)$                                | cross polytope                      |
| 2    | $2k$ | $(2k - 4, 2k - 2)$                      | 2$k$-gon                            |
| 3    | 12   | $(2, 4)$                                | icosahedron                         |
| 7    | 56   | $(2, 4)$                                | kissing configuration               |
| 8    | 240  | $(4, 6)$                                | $E_8$ roots                         |
| 23   | 552  | $(2, 4)$                                | equiangular lines                   |
| 23   | 4600 | $(4, 6)$                                | kissing configuration               |
| 24   | 196560 | $(8, 10)$                             | Leech lattice                       |
Let $C \subset \mathbb{R}^d$ be a convex body,

$$\sigma_C(B) = |\{x \in \partial C : n_x \in B\}|_{d-1}$$

for all Borel $B \subseteq S^{d-1}$, and $h_C$ be the support function of $C$

$$h_C(y) = \sup_{x \in C} \langle x, y \rangle.$$

Given two convex bodies, $C$ and $D$, and $p \geq 1$, we define the $L^p$-mixed volume of the two to be

$$V_p(C, D) = \frac{p}{d} \lim_{\varepsilon \to 0} \frac{|C + \varepsilon D|_d - |C|_d}{\varepsilon} = \frac{1}{d} \int_{S^{d-1}} h_D(x)^p h_C(x)^{1-p} d\sigma_C(x),$$

where $C + \varepsilon D$ is the convex body with support function

$$h_{C + \varepsilon D}(x) = \sqrt[p]{h_C(x)^p + \varepsilon h_D(x)^p}.$$
The $L^p$-projection body of $C$, $\Pi_p C$, is the origin-symmetric convex body with support function

$$h_{\Pi_p C}(x) = \left( c_{d,p} \int_{S^{d-1}} |\langle x, y \rangle|^p h_C(x)^{1-p} d\sigma_C(x) \right)^{\frac{1}{p}}.$$  

Defining $\sigma_{C,p}$ such that $d\sigma_{C,p}(x) = h_C(x)^{1-p} d\sigma_C(x)$, we see that

$$I_{F_p}(\sigma_{C,p}) = \int\int_{S^{d-1} \times S^{d-1}} |\langle x, y \rangle|^p d\sigma_{C,p}(x) d\sigma_{C,p}(y)$$

$$= \frac{1}{c_{d,p}} \int_{S^{d-1}} h_{\Pi_p C}(x)^p h_C(x)^{1-p} d\sigma_C(x) = \frac{d}{c_{d,p}} V_p(C, \Pi_p C).$$

Thus, minimizing the $p$-frame energy (over admissible measures) is the same as minimizing $V_p(C, \Pi_p C)$ over all symmetric convex bodies $C$ (scaled to satisfy $\sigma_{C,p}(S^{d-1}) = 1$).

**Proposition (Bilyk, Glazyrin, Matzke, Park, Vlasiuk, 2022)**

The quantity $\frac{V_1(C, \Pi_1 C)}{|\partial C|^2_{d-1}}$ is minimized if and only if $C$ is a hypercube.
Thank you!

This work was in collaboration with Dmitriy Bilyk, Alexey Glazyrin, Josiah Park, and Oleksandr Vlasiuk, and was supported in part by the National Science Foundation Graduate Research Fellowship Grant 00039202.