A zoo of dualities

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joint work with S. Artstein-Avidan and S. Sadovsky

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Our favourite duality

The polarity transform, $^\circ : \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$, is given by

$$K^\circ = \{y \in \mathbb{R}^n : \forall x \in K,\ \langle x, y\rangle \le 1\}.$$

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- The only invariant set is $B_2^n = \{x \in \mathbb{R}^n :\ |x| \le 1\}$.
- Böröczky and Schneider showed that polarity is essentially the only order reversing involution on $\mathcal{K}_0^n$.
- Blaschke-Santaló inequality: for centrally symmetric $K \in \mathcal{K}_0^n$ we have
$$\mathrm{Vol}(K)\mathrm{Vol}(K^\circ) \le \mathrm{Vol}(B_2^n)^2$$

Definition

Let X be a set, and let $T : \mathcal{P}(X) \to \mathcal{P}(X)$, where by $\mathcal{P}(X)$ we denote the power set of X. The map T is an **order reversing quasi involution** *if for every $K, L \subseteq X$, the following hold*

1. $K \subseteq TTK$, *(quasi involution)*
2. *if* $L \subseteq K$ *then* $TK \subseteq TL$. *(order reversion)*

Let $\mathcal{C}$ be the image of T, i.e. $\mathcal{C} = \{K \subset X : \exists L \subset X \text{ s.th. } K = TL\}$.

We say that $T|_{\mathcal{C}}$ is a **duality** (order reversing involution).

Elementary properties

Let $T : \mathcal{P}(X) \to \mathcal{P}(X)$ be an order reversing quasi involution. Then $TTX = X$ and $T\emptyset = X$ and for any collection of sets $K_i \subseteq X$, $i \in I$,

$$T\left(\cup_{i\in I} K_i\right) = \cap_{i\in I} T(K_i).$$

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Proposition

Let $T : \mathcal{P}(X) \to \mathcal{P}(X)$ be an order reversing quasi involution, and let $K \subseteq X$. Then

$$TTK = \cap\{L : L \supseteq K \quad \text{and} \quad L = TTL\}.$$

This means that for any set $K \subseteq X$, the set TTK is the "envelope" of K, namely the smallest set in the image of T which contains K.

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Definition

Let X be some set, $\mathcal{C} \subseteq \mathcal{P}(X)$ and $T : \mathcal{C} \to \mathcal{C}$. We say that the map T **respects inclusions** *if $L \subseteq \cup_{i \in I} K_i$ implies $TL \supseteq \cap_{i \in I} TK_i$ for any $L, K_i \in \mathcal{C}$, $i \in I$.*

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Theorem (Artstein-Avidan, Sadovsky, W.)

Let $\mathcal{C} \subseteq \mathcal{P}(X)$ be a family of sets and $T : \mathcal{C} \to \mathcal{C}$ be an order reversing quasi involution on $\mathcal{C}$ which respects inclusions. Then T can be extended to an order reversing quasi involution $\hat{T} : \mathcal{P}(X) \to \mathcal{P}(X)$ with $\hat{T}|_{\mathcal{C}} = T$.

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Definition (Cost duality)

Let $c : X \times X \to (-\infty, \infty]$ *satisfy* $c(x, y) = c(y, x)$*. For* $K \subseteq X$ *define the* c-**dual** *set of* K *as*

$$K^c = \{y \in X : \forall x \in K,\ c(x, y) \geq 0\}.$$

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Theorem (Artstein-Avidan, Sadovsky, W.)

Let $T : \mathcal{P}(X) \to \mathcal{P}(X)$ *be an order reversing quasi involution. Then there exists a cost function* $c : X \times X \to \{\pm 1\}$ *such that for all* $K \subseteq X$ *we have* $TK = K^c$.

Proof

Invariant sets: $K = TK$

Fact

Let $T : \mathcal{P}(X) \to \mathcal{P}(X)$ *be an order reversing quasi involution. If* $K = TK$ *then* $K \subseteq X_0 = \{x : c(x,x) \geq 0\} = \{x : x \in T(\{x\})\}$.

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Lemma

Let $T : \mathcal{P}(X) \to \mathcal{P}(X)$ be an order reversing quasi involution and denote X_0 as above.

1. *If $TX_0 = X_0$ then X_0 is the unique invariant set for the transform.*
2. *If $TX_0 \not\subseteq X_0$ then there is no invariant set for the transform.*
3. *If $TX_0 \subsetneq X_0$ then there are examples where no invariant set exists, examples where only one invariant set exists, and examples where more than one invariant set exists.*

A zoo of Examples

Polarity once again

Consider the polarity transform $T : \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ given by

$$TK = K^\circ = \{y : \forall x \in K,\ \langle x, y\rangle \le 1\}.$$

- The associated set is $S = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n :\ \langle x, y\rangle \le 1\}$.
- To write it as a cost-transform, one may take $c(x, y) = -\langle x, y\rangle + 1$ so that
  $$K^c = \{y : \forall x \in K,\ -\langle x, y\rangle + 1 \ge 0\} = K^\circ.$$
- $X_0 = \{x :\ \langle x, x\rangle \le 1\}$ and $TX_0 = X_0$, hence it is the unique invariant set.

Legendre transform

Consider the transform $T : \mathcal{P}(\mathbb{R}^{n+1}) \to \mathcal{P}(\mathbb{R}^{n+1})$ defined by

$$T(\operatorname{epi}\varphi) = \operatorname{epi}(\mathcal{L}\varphi),$$

where $\mathcal{L}$ denotes the Legendre transform

$$\mathcal{L}\varphi(y) = \sup_x \left(\langle x, y\rangle - \varphi(x)\right).$$

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$$\mathcal{L}\varphi(y) = \sup_x \left(\langle x, y\rangle - \varphi(x)\right).$$

- The associated set is

$$S = \{((x,t),(y,s)) : \langle x, y\rangle \le s + t\}.$$

- The image class for this transform is the class of epi-graphs of functions in $\mathrm{Cvx}(\mathbb{R}^n)$ together with the constant $+\infty$ and the constant $-\infty$ functions.
- To write it as a cost transform, one may take

$$c((x,t),(y,s)) = t + s - \langle x, y\rangle.$$

- The only invariant set is $\operatorname{epi}(\|x\|_2^2/2)$.

Complements of neighborhoods

Consider the transform $T : \mathcal{P}(X) \to \mathcal{P}(X)$ where (X, d) is a metric space, given by

$$TA = \{y \in X :\ \forall x \in A,\ d(x, y) \geq \varepsilon\},$$

which maps a set to the complement of its ε-neighborhood.

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- The associated set is

$$S = \{(x, y) : d(x, y) \geq \varepsilon\}.$$

- The image class for this transform consists of complements of unions of ε-balls. For example, all convex sets are in the class.
- To write T as a cost transform, one may take $c(x, y) = d(x, y) - \varepsilon$.
- Clearly there are no invariant sets.

Producing new dualities

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Definition

Given a topological space X and an order reversing quasi involution $T : \mathcal{P}(X) \to \mathcal{P}(X)$ with an associated set

$$S_T = \{TK \times TTK : K \subseteq X\} \subset X \times X$$

we define its **dual order reversing quasi involution** *to be $T' : \mathcal{P}(X) \to \mathcal{P}(X)$ with an associated set $S_{T'} = \overline{X \times X \setminus S_T}$.*

Ball intersections

Let (X, d) be some metric space. Let

$$S = \{(x, y) \in X \times X : \ d(x, y) \leq \varepsilon\}.$$

The associated transform is given by

$$TA = \cap_{x \in A} B(x, \varepsilon).$$

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- The image class consists of all sets obtained by intersections of balls of radius ε. In particular, these sets are closed and of diameter at most 2ε.
- The invariant sets are the so-called "diametrically complete" sets, and when $X = \mathbb{R}^n$ with the Euclidean distance d, these are precisely sets of equal width ε.

Dual polarity

Let

$$S = \{(x,y) :\ \langle x, y\rangle \geq 1\} \subseteq \mathbb{R}^n \times \mathbb{R}^n$$

The associated transform is given by

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Dual polarity

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$$TA = \{y : \forall x \in A, \ \langle x, y\rangle \geq 1\}.$$

The image class consists of intersections of affine half-spaces that do not include the origin. In particular, these are unbounded, closed and convex sets.

Lemma

The class $\mathcal{C} = \{TK : K \subseteq \mathbb{R}^n\}$ consists of $\mathbb{R}^n$ together with all closed convex sets $K \subseteq \mathbb{R}^n$ that do not include the origin and satisfy for all $\lambda \geq 1$ that $\lambda K \subseteq K$.

The class $\mathcal{C}$ decomposes into sub-classes:

For every $u \in S^{n-1}$ we define the sub-class $\mathcal{C}_u$ to be those $K \in \mathcal{C}$ whose closest point to the origin lies on the ray $u\mathbb{R}^+$.

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$$\mathcal{C}_u = \cup_{a>0} \mathcal{C}_{u,a},$$

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Therefore, having fixed an orthonormal basis $\{e_i\}_{i=1}^n$ for $\mathbb{R}^n$, in order to study T it suffices to focus on one sub-class $\mathcal{C}_{e_n,1}$.

More about the subclass $C_{e_n,1}$

Lemma

For $K \in \mathcal{C}_{e_n,1} \subset \mathcal{P}(\mathbb{R}^n)$ *we have that*

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Let J be a transform on subsets of $\mathbb{R}^{n-1} \times \mathbb{R}^+$ defined by $JK = F(K)$, where

$$F(x,t) = (x/t, 1/t).$$

This is a convexity preserving map that maps rays emanating from the origin to rays parallel to the ray $\{0\} \times \mathbb{R}^+$.

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Let $\tilde{J}(K) = JK \cup R_{e_n^\perp} JK$

The resulting body always includes the segment $[-e_n, e_n]$, is included in the slab $\{|\langle \cdot, e_n \rangle| \leq 1\}$, and is invariant under reflections about $e_n^\perp$.

Lemma

For $K \in \mathcal{C}_{e_n,1} \subset \mathcal{P}(\mathbb{R}^n)$ *we have that*

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Theorem (Artstein-Avidan, Sadovsky, W.)

Let $K \subset \mathcal{C}$ be essentially symmetric. Let $T : \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ be given by $TK = \{x : \forall y \in K,\ \langle x, y\rangle \geq 1\}$. Then

$$\gamma_n(K)\gamma_n(TK) \leq \gamma_n(K_0)^2,$$

where $K_0 = \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}^+ : |x|^2 + 1 \leq t^2\}$, and γ_n is the Gaussian measure on $\mathbb{R}^n$.

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Lemma. Let $K \in \mathcal{C}_{e_n,1} \subset \mathcal{P}(\mathbb{R}^n)$. Then $\gamma_n(K) = \nu(JK)$, where

$$d\nu(x,z) = (2\pi)^{-n} e^{-|x|^2/2z^2} e^{-1/2z^2} z^{-(n+1)} dx dz$$

is defined on $\mathbb{R}^{n-1} \times \mathbb{R}^+$ (and $d\nu(x,z)$ is 0 for $z \leq 0$).

Equivalently

Theorem

Let $L \subseteq \mathbb{R}^n$ be a centrally symmetric convex body which includes the segment $[-e_n, e_n]$, is included in the slab $\{|\langle \cdot, e_n \rangle| \leq 1\}$, and is invariant to reflections about $e_n^\perp$. Then

$$\nu(L)\nu(L^\circ) \leq \nu(B_2^n)^2,$$

where $d\nu(x, z) = (2\pi)^{-n} e^{-|x|^2/2z^2} e^{-1/2z^2} z^{-(n+1)} dxdz$ on $\mathbb{R}^{n-1} \times \mathbb{R}^+$.

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Sketch of the proof

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Lemma (D. Cordero-Erausquin)

For a centrally symmetric convex set $L \subseteq \mathbb{R}^n$ we have

$$\gamma_n(L)\gamma_n(L^\circ) \leq \gamma_n(B_2^n)^2.$$

Moreover, for any $\alpha > 0$ we have that $\gamma_n(\alpha L)\gamma_n(\alpha L^\circ) \leq \gamma_n(\alpha B_2^n)^2$.

Thank you!