On discrete Brunn-Minkowski type inequalities

Asymptotic Geometric Analysis Seminar
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The Brunn-Minkowski inequality

Theorem

Given $K, L \in \mathcal{K}^n$ we have $\mathrm{vol}(K + L)^{1/n} \geq \mathrm{vol}(K)^{1/n} + \mathrm{vol}(L)^{1/n}$.

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Naturally, due to the homogeneity of vol of degree n, for any $\lambda, \mu \geq 0$

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As a consequence of the (weighed) arithmetic and geometric means inequality, we can obtain:

Corollary

Given $K, L \in \mathcal{K}^n$ we have $\mathrm{vol}\big((1-\lambda)K + \lambda L\big) \geq \mathrm{vol}(K)^{1-\lambda}\mathrm{vol}(L)^{\lambda}$ for any $\lambda \in (0,1)$.

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Note: The inequality can be extended to arbitrary non-empty compact sets, and even to more general measurable families.

Functional counterpart:

Theorem: The Prékopa-Leindler inequality

Let $\lambda \in (0,1)$ and let $f, g, h : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be Lebesgue integrable functions verifying

$$h\big((1-\lambda)x + \lambda y\big) \geq f(x)^{1-\lambda} g(y)^{\lambda}$$

for all $x, y \in \mathbb{R}^n$.

Related inequalities

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for all $x, y \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} h(x)dx \geq \left(\int_{\mathbb{R}^n} f(x)dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(x)dx\right)^{\lambda}.$$

Related inequalities

Generalization of Prékopa-Leindler's:

Theorem: The Borell-Brascamp-Lieb inequality

Let $\lambda \in (0,1)$, let $-1/n \leq \alpha \leq \infty$ and let $f, g, h : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be Lebesgue integrable functions verifying

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Brunn-Minkowski generalizations

In general $h_{K+L} = h_K + h_L$.

Definition (Firey (1962))

Let $p \geq 1$ and $K, L \in \mathcal{K}^n$ containing the origin in their interior. Then the p-sum $K +_p L$ is the unique convex body such that

$$h_{K+_pL} = \left(h_K^p + h_L^p\right)^{1/p}.$$

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Definition (Lutwak, Yang, Zhang (2012))

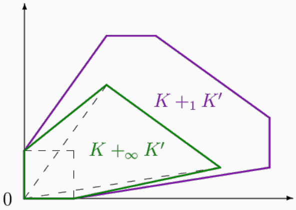
Let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets and let $p \geq 1$. Then

$$K +_p L = \left\{(1-\mu)^{1/q}x + \mu^{1/q}y : x \in K, y \in L, \ \mu \in [0,1]\right\},$$

where $q \in [1, +\infty]$ is the Hölder conjugate of p, i.e., such that $1/p + 1/q = 1$.

Brunn-Minkowski generalizations

- $p = 1$: $K +_1 L = K + L$ (Minkowski addition).
- $p = \infty$: $K +_\infty L = \text{conv}(K \cup L)$ (convex hull).
- If $p \leq q$ then:
  - $K +_q L \subset K +_p L$.
  - $(1 - \lambda) \cdot K +_p \lambda \cdot L \subset (1 - \lambda) \cdot K +_q \lambda \cdot L$.



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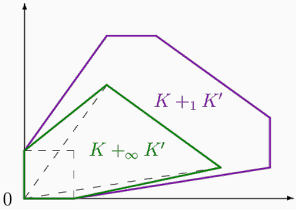

Theorem (Firey (1962), Lutwak, Yang, Zhang (2012))

Let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets, and let $p \geq 1$. Then

$$\mathrm{vol}(K +_p L)^{p/n} \geq \mathrm{vol}(K)^{p/n} + \mathrm{vol}(L)^{p/n}.$$

Brunn-Minkowski generalizations

Definition

Given $f : \mathbb{S}^{n-1} \to \mathbb{R}_{\geq 0}$, the Wulff shape of f is

$$W(f) = \bigcap_{u \in \mathbb{S}^{n-1}} \{x \in \mathbb{R}^n : \langle x, u \rangle \leq f(u)\} .$$

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Then, for any $K \in \mathcal{K}^n$ containing the origin, $K = W(h_K)$. Thus, for any $K, L \in \mathcal{K}^n$ containing the origin and any $p \geq 1$,

$$(1-\lambda) \cdot K +_p \lambda \cdot L = W\big(((1-\lambda)h_K^p + \lambda h_L^p)^{1/p}\big).$$

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This definition can now be extended to $0 \leq p < 1$, in particular,

$$(1-\lambda) \cdot K +_0 \lambda \cdot L = W\big(h_K^{1-\lambda} h_L^{\lambda}\big).$$

Brunn-Minkowski generalizations

Böröczky, Lutwak, Yang and Zhang conjectured:

Conjecture - The log-Brunn-Minkowski inequality

Let $K, L \subset \mathbb{R}^n$ be centrally symmetric convex bodies, and let $\lambda \in (0, 1)$. Then

$$\mathrm{vol}\big((1-\lambda)\cdot K +_0 \lambda \cdot L\big) \geq \mathrm{vol}(K)^{1-\lambda}\mathrm{vol}(L)^{\lambda}.$$

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- $n = 2$ (Böröczky, Lutwak, Yang, Zhang, 2012)
- Unconditional bodies for $p = 0$ (Saroglou, 2015)
- Unconditional bodies for $0 < p < 1$ (Marsiglietti, 2015)
- Symmetric w.r.t. n independent hyperplanes (Böröczky, Kalantzopoulos, 2020)

Discretization preliminaries

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The objects of study in this setting can be

- Discrete sets $A \subset \mathcal{L} \longrightarrow$ Cardinality $|A|$.
- Convex bodies $K \subset \mathcal{K}^n \longrightarrow$ Lattice point enumerator $G(K) = |K \cap \mathcal{L}|$.

Discreticing Brunn-Minkowski for the cardinality

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- Ruzsa (1994): $|A+B| \geq |A| + n|B| - \frac{n(n+1)}{2}$ when $|B| \leq |A|$ and $\dim(A+B) = n$.

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$$|\bar{A} + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}.$$

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- Iglesias, Yepes Nicolás & Zvavitch (2020):

$$|A + B + \{0,1\}^n|^{1/n} \geq |A|^{1/n} + |B|^{1/n}.$$

Discreticing Brunn-Minkowski for $G(K)$

Theorem (Iglesias, Yepes Nicolás, Zvavitch (2020))

Let K, L be non-empty bounded sets and let $\lambda \in (0, 1)$. Then

$$G((1-\lambda)K + \lambda L + (-1,1)^n)^{1/n} \geq (1-\lambda)G(K)^{1/n} + \lambda G(L)^{1/n}.$$

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When $\lambda = 1/2$, one may obtain other discrete analogues:

Let K, L be non-empty bounded sets. Then

$G\left(\frac{K+L}{2} + [0,1]^n\right) \geq \sqrt{G(K)G(L)}$ (Halikias, Klartag & Slomka)

$G\left(\frac{K+L}{2} + [0,1]^n\right) \geq \frac{G(K)^{1/n}+G(L)^{1/n}}{2}$ (Iglesias, Yepes Nicolás & Zvavitch)

A discrete Borell-Brascamp-Lieb inequality

Theorem (Iglesias, Yepes Nicolás, Zvavitch (2020))

Let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets, $\lambda \in (0,1)$ and $-1/n \leq \alpha \leq \infty$. Let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be such that

$$h\big((1-\lambda)x + \lambda y\big) \geq \mathcal{M}_\alpha^\lambda\big(f(x), g(y)\big)$$

for all $x \in K$, $y \in L$.

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for all $x \in K$, $y \in L$. Then

$$\sum_{z \in (M + (-1,1)^n) \cap \mathbb{Z}^n} h^\diamond(z) \geq \mathcal{M}^\lambda_{\frac{\alpha}{n\alpha+1}}\left(\sum_{x \in K \cap \mathbb{Z}^n} f(x), \sum_{y \in L \cap \mathbb{Z}^n} g(y)\right),$$

where $M = (1-\lambda)K + \lambda L$ and $h^\diamond(z) = \sup_{u \in (-1,1)^n} h(z+u)$ for all $z \in \mathbb{R}^n$.

Discrete analogues for general coefficients

Theorem (Iglesias, L., Yepes Nicolás (2020))

Let $t, s > 0$ and let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets. Let $-1/n \leq \alpha \leq \infty$, $\alpha \neq 0$, and let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be non-negative functions such that

$$h(tx + sy) \geq \left[tf(x)^\alpha + sg(y)^\alpha\right]^{1/\alpha}$$

for all $x \in K$, $y \in L$ with $f(x)g(y) > 0$.

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for all $x \in K$, $y \in L$ with $f(x)g(y) > 0$. Then

$$\sum_{z \in \left(M + (-1, \lceil t+s \rceil)^n\right) \cap \mathbb{Z}^n} h^{\bar{\diamond}}(z) \geq \mathcal{S}^{t,s}_{\frac{\alpha}{n\alpha+1}} \left(\sum_{x \in K \cap \mathbb{Z}^n} f(x), \sum_{y \in L \cap \mathbb{Z}^n} g(y) \right),$$

where $M = tK + sL$ and $h^{\bar{\diamond}}(z) = \sup_{u \in (-1, \lceil t+s \rceil)^n} h(z - u)$.

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Let $t, s \geq 0$ and let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets such that $G(K)G(L) > 0$. Then

$$G(tK + sL + (-1, \lceil t+s \rceil)^n)^{1/n} \geq tG(K)^{1/n} + sG(L)^{1/n}. \quad (1)$$

The inequality is sharp.

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$$\mathrm{vol}(tK + sL)^{1/n} \geq t\mathrm{vol}(K)^{1/n} + s\mathrm{vol}(L)^{1/n},$$

that is, the classical Brunn-Minkowski inequality.

A BBL type inequality for negative coefficients

Theorem (Dancs, Uhrin (1980))

Let $\lambda \in (0,1)$ and $-\infty \leq \alpha \leq -1/n$. Let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be integrable functions such that

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A discrete version for negative coefficients

Theorem (L. (2022))

Let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets, $\lambda \in (0,1)$ and $-\infty \leq \alpha \leq -1/n$. Let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be such that

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$$\sum_{z \in (M + (-1,1)^n) \cap \mathbb{Z}^n} h^\diamond(z) \geq \min\left\{(1-\lambda)^{n+1/\alpha} \sum_{x \in K \cap \mathbb{Z}^n} f(x),\ \lambda^{n+1/\alpha} \sum_{y \in L \cap \mathbb{Z}^n} g(y)\right\},$$

where $M = (1-\lambda)K + \lambda L$.

Discrete analogues in the L_p setting

Theorem (Hernández Cifre, L., Yepes Nicolás (2021))

Let $\lambda \in (0,1)$ and $p \geq 1$, and let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets. Let $-1/n \leq \alpha \leq \infty$ and let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be non-negative functions such that

$$h\Big((1-\lambda)^{1/p}(1-\mu)^{1/q}x + \lambda^{1/p}\mu^{1/q}y\Big) \geq \Big[(1-\lambda)^{1/p}(1-\mu)^{1/q}f(x)^{\alpha} + \lambda^{1/p}\mu^{1/q}g(y)^{\alpha}\Big]^{1/\alpha}$$

for all $x \in K$, $y \in L$ with $f(x)g(y) > 0$ and all $\mu \in [0,1]$.

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for all $x \in K$, $y \in L$ with $f(x)g(y) > 0$ and all $\mu \in [0,1]$. Then

$$\sum_{z \in (M_p + (-1,1)^n) \cap \mathbb{Z}^n} h^\diamond(z) \geq \mathcal{M}^\lambda_{\frac{p\alpha}{n\alpha+1}}\left(\sum_{x \in K \cap \mathbb{Z}^n} f(x), \sum_{y \in L \cap \mathbb{Z}^n} g(y)\right),$$

where $M_p = (1-\lambda) \cdot K +_p \lambda \cdot L$.

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$$G\big((1-\lambda)\cdot K +_p \lambda \cdot L + (-1,1)^n\big)^{p/n} \geq (1-\lambda)G(K)^{p/n} + \lambda G(L)^{p/n}. \quad (2)$$

The inequality is sharp and the cube cannot be reduced.

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Theorem (Hernández Cifre, L., Yepes Nicolás (2021))

Let $\lambda \in (0, 1)$ and $p \geq 1$, and let $K, L \subset \mathbb{R}^n$ be compact sets with $G(K)G(L) > 0$. Then (2) implies

$$\mathrm{vol}((1-\lambda)\cdot K +_p \lambda \cdot L)^{p/n} \geq (1-\lambda)\mathrm{vol}(K)^{p/n} + \lambda \mathrm{vol}(L)^{p/n},$$

that is, the continuous L_p Brunn-Minkowski inequality.

Discrete analogues in the L_0 setting

Some useful relations:

$$G(K) \leq \mathrm{vol}\left(K + \left(-\frac{1}{2}, \frac{1}{2}\right)^n\right),$$
$$\mathrm{vol}(K) \leq G\left(K + \left(-\frac{1}{2}, \frac{1}{2}\right)^n\right).$$

Discrete analogues in the L_0 setting

Theorem (Hernández Cifre, L. (2021))

Let $K, L \subset \mathbb{R}^n$ be centrally symmetric convex bodies and let $\lambda \in (0,1)$. If either K, L are unconditional convex bodies or $n = 2$, then

$$G\left((1-\lambda)\cdot\left(K+\left[-\frac{1}{2},\frac{1}{2}\right]^n\right)+_0\lambda\cdot\left(L+\left[-\frac{1}{2},\frac{1}{2}\right]^n\right)+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right) \geq G(K)^{1-\lambda}G(L)^{\lambda}.$$

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- The cubes cannot be reduced.
- It implies $\mathrm{vol}\big((1-\lambda)\cdot K +_0 \lambda\cdot L\big) \geq \mathrm{vol}(K)^{1-\lambda}\mathrm{vol}(L)^{\lambda}$, that is, the log-Brunn-Minkowski inequality, for both unconditional convex bodies or when $n = 2$.

Discrete analogues in the L_0 setting

Theorem (Hernández Cifre, L. (2021))

Let $K, L \subset \mathbb{R}^n$ be two unconditional convex bodies and let $\lambda \in (0,1)$. Then, for any $0 < p < 1$,

$$G\left((1-\lambda)\cdot\left(K+\left[-\frac{1}{2},\frac{1}{2}\right]^n\right) +_p \lambda\cdot\left(L+\left[-\frac{1}{2},\frac{1}{2}\right]^n\right) + \left(-\frac{1}{2},\frac{1}{2}\right)^n\right)$$
$$\geq \mathcal{M}^{\lambda}_{p/n}\big(G(K), G(L)\big).$$

Furthermore, it implies the L_p Brunn-Minkowski inequality for $0 < p < 1$ for unconditional convex bodies.

Rogers-Shephard inequalities

In the particular case when $L = -K$ and $\lambda = 1/2$, the Brunn-Minkowski inequality gives

$$\mathrm{vol}(K - K) \geq 2^n \mathrm{vol}(K).$$

Rogers-Shephard inequalities

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An upper bound is given by the *Rogers-Shephard inequality*:

Theorem (The Rogers-Shephard inequality)

Let $K \in \mathcal{K}^n$. Then

$$\mathrm{vol}(K - K) \leq \binom{2n}{n} \mathrm{vol}(K).$$

Rogers-Shephard inequalities

This relation can be generalized for two convex bodies $K, L \in \mathcal{K}^n$:

$$\mathrm{vol}(K+L)\mathrm{vol}\big(K\cap(-L)\big) \leq \binom{2n}{n}\mathrm{vol}(K)\mathrm{vol}(L).$$

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Rogers and Shephard also gave the following projection-section bound:

Theorem

Let $k \in \{1, \dots, n-1\}$ and $H \in \mathcal{L}_k^n$. Let $K \in \mathcal{K}^n$ be a convex body. Then

$$\mathrm{vol}_{n-k}(P_{H^\perp}K)\mathrm{vol}_k(K\cap H) \leq \binom{n}{k}\mathrm{vol}(K).$$

Rogers-Shephard inequalities

Interestingly, both theorems follow from a classical result due to Berwald:

Theorem (Berwald's inequality)

Let $K \in \mathcal{K}^n$ be a convex body with $\dim K = n$ and let $f : K \longrightarrow \mathbb{R}_{\geq 0}$ be a concave function. Then, for any $0 < p < q$,

$$\left(\frac{\binom{n+q}{n}}{\mathrm{vol}(K)} \int_K f^q(x)\,\mathrm{d}x \right)^{1/q} \leq \left(\frac{\binom{n+p}{n}}{\mathrm{vol}(K)} \int_K f^p(x)\,\mathrm{d}x \right)^{1/p}.$$

Discrete Rogers-Shephard analogues

One cannot expect to obtain a discrete analogue of the form

$$G(K - K) \leq \binom{2n}{n} G(K).$$

Indeed, $K = [-1/2, 1/2]^n$ for $n < 5$ is a counterexample.

Discrete Rogers-Shephard analogues

One cannot expect to obtain a discrete analogue of the form

$$G(K-K) \leq \binom{2n}{n} G(K).$$

Indeed, $K = [-1/2, 1/2]^n$ for $n < 5$ is a counterexample. However, we can modify the upper bound by adding a cube and obtain:

Theorem (Alonso-Gutiérrez, L., Yepes Nicolás (2021))

Let $K \subset \mathbb{R}^n$ be a non-empty convex bounded set. Then

$$G(K-K) \leq \binom{2n}{n} G\left(K + \left(-\frac{3}{4}, \frac{3}{4}\right)^n\right).$$

Discrete Rogers-Shephard analogues

Theorem (Alonso-Gutiérrez, L., Yepes Nicolás (2021))

Let $K, L \subset \mathbb{R}^n$ be non-empty convex bounded sets. Then

$$G(K+L)G\big(K\cap(-L)\big) \leq \binom{2n}{n} G\big(K+(-1,1)^n\big)G\big(L+(-1,1)^n\big).$$

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Theorem (Alonso-Gutiérrez, L., Yepes Nicolás (2021))

Let $K, L \subset \mathbb{R}^n$ be non-empty convex bounded sets. Then

$$G(K+L)G\big(K\cap(-L)\big) \leq \binom{2n}{n} G\big(K+(-1,1)^n\big)G\big(L+(-1,1)^n\big).$$

In particular, taking $L=-K$, with $0 \in K$,

$$G(K-K) \leq \binom{2n}{n} \frac{G\big(K+(-1,1)^n\big)^2}{G(K)}.$$

Discrete Rogers-Shephard analogues

The projection-section theorem also admits a similar discrete analogue:

Theorem (Alonso-Gutiérrez, L., Yepes Nicolás (2021))

Let $k \in \{1, \ldots, n-1\}$ and $H = \operatorname{lin}\{e_1, \ldots, e_k\} \in \mathcal{L}^n_k$. Let $K \subset \mathbb{R}^n$ be a non-empty convex bounded set. Then

$$G_{n-k}(P_{H^\perp} K) G_k(K \cap H) \leq \binom{n}{k} G\big(K + (-1, 1)^n\big).$$

Discrete Rogers-Shephard analogues

Finally, a discrete analogue of Berwald's inequality can also be obtained:

Theorem (Alonso-Gutiérrez, L., Yepes Nicolás (2021))

Let $K \subset \mathbb{R}^n$ be a convex bounded set containing the origin and let $f : K \longrightarrow \mathbb{R}_{\geq 0}$ be a concave function with $f(0) = |f|_\infty$. Then, for any $0 < p < q$,

$$\left(\frac{\binom{n+q}{n}}{G(K)} \sum_{x \in K \cap \mathbb{Z}^n} f^q(x) \right)^{1/q} \leq \left(\frac{\binom{n+p}{n}}{G(K)} \sum_{x \in (K + (-1,1)^n) \cap \mathbb{Z}^n} (f^\diamond)^p(x) \right)^{1/p}.$$

Discrete Rogers-Shephard analogues

Finally, a discrete analogue of Berwald's inequality can also be obtained:

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From this, alternative discrete Rogers-Shephard type inequalities can be derived.

On discrete Brunn-Minkowski type inequalities

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