Around Bezout inequalities for mixed volumes

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October 2022

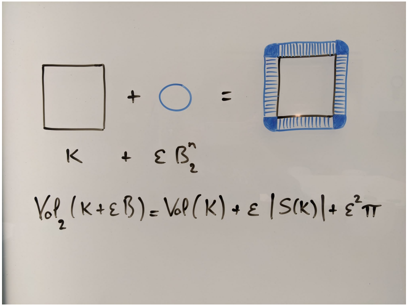
AGA Seminar - Zoom

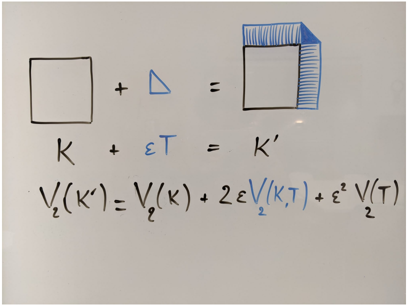
Mixed volume : Minkowski's definition

Denote by $\mathcal{K}_n = \{K \subset \mathbb{R}^n : K \text{ compact convex set}\}$.

Steiner polynomial

For any $K \in \mathcal{K}_n$, there exists non-negative $\{v_k; 0 \leq k \leq n\}$ such that : $\forall \varepsilon > 0$, $Vol_n(K + \varepsilon B_2^n) = \sum_{k=0}^{n} \binom{n}{k} v_k \varepsilon^k$.





$Vol_n(K + \varepsilon T)$ is also a polynomial in ε (even if $T \in \mathcal{K}_n$ is not a ball).

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Let $K, L \in \mathcal{K}_n$, let $\lambda, \mu \geq 0$. Then $Vol_n(\lambda K + \mu L)$ is a polynomial in (λ, μ) :

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where $v_k = V_n(K[k], L[n-k]) = V_n(K, ..., K, L, ..., L)$ are called mixed volumes.

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- Let $K_1, ..., K_m \in \mathcal{K}_n$. Let $\lambda_1, ..., \lambda_m \geq 0$. Then :

$$Vol_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{\substack{a=(a_1,\ldots,a_m)\\ |a|=n}} \binom{n}{a} v_a \;\; {\lambda_1}^{a_1} ... {\lambda_m}^{a_m}$$

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  where $v_a = V_n(K_1[a_1], \ldots, K_m[a_m])$ are called mixed volumes.
- $V_n : \mathcal{K}_n^n \to [0, +\infty)$ is a multilinear, continuous functional.

  Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transform. Then :

$$V_n(TK_1, ..., TK_n) = |det(T)| V_n(K_1, ..., K_n)$$

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- $V(K[n-1], [0, u], [0, v]) = \frac{1}{n(n-1)} Vol_2([0, u] + [0, v]) Vol_{n-2}\left(\pi_{(u,v)^\perp}(K)\right)$.

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Assume P is a polytope : $P = \cap_{i=1}^{N} H^{-}(u_i, h_i)$.
where $H^{-}(u, b) := \{x \in \mathbb{R}^n : \langle x, u \rangle \leq b\}$

Mixed volume and surface area

Assume P is a polytope : $P = \cap_{i=1}^{N} H^{-}(u_i, h_i)$.

Then for any convex body L :

$$V(L, P[n-1]) = \frac{1}{n} \sum_{i=1}^{N} h_L(u_i) Vol_{n-1}(P^{u_i})$$

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For any convex body K, there exists a (finite, non-negative) measure S_K on $\mathbb{S}^{n-1}$, such that :

$$\text{(for any } L\text{)}, \qquad V(L, K[n-1]) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_L(u)dS_K(u).$$

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Mixed volume and (mixed) surface areas

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S_K is a discrete measure with finite support, if and only if K is a polytope.

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More generally, if $K_1, ..., K_{n-1}$ are convex bodies, then there exists a measure σ on $\mathbb{S}^{n-1}$, such that :

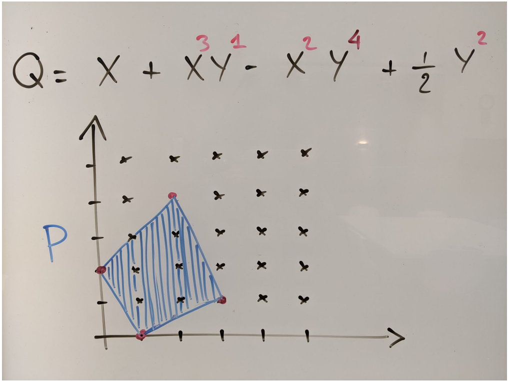
$$\text{(for any } L\text{)}, \qquad V(L, K_1, ..., K_{n-1}) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L(u) d\sigma(u).$$

The measure σ is called a mixed surface area measure, and is usually denoted :

$$\sigma = S(K_1, ..., K_{n-1}, .)$$

Newton polytope of a polynomial

Let $Q(X,Y) = \sum_\alpha c_\alpha X^{\alpha_1} Y^{\alpha_2} \in \mathbb{R}[X,Y]$. Then the Newton polytope of Q is $P = Conv\{(\alpha_1, \alpha_2) : c_\alpha \neq 0\}$.



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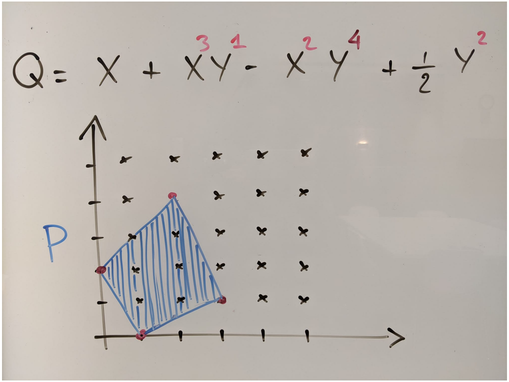
More generally, if $Q \in \mathbb{R}[X_1, ..., X_n]$, $Q = \sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha$,
then the Newton polytope of Q is $P = Conv\{\alpha : c_\alpha \neq 0\} \subset \mathbb{R}^n_+$.



Let $f_1, ..., f_r : \mathbb{R}^n \to \mathbb{R}$ be polynomials. Denote by $X_1, ..., X_r$ the associated algebraic varieties $(X_i := \{x \in \mathbb{R}^n : f_i(x) = 0\})$.

The *Bezout inequality* states that :

$$deg(X_1 \cap ... \cap X_r) \leq \prod deg(X_i) \qquad [B]$$

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We can reformulate [B] within the language of mixed volumes :

$$V(P_1, ..., P_r, \Delta[n-r])V(\Delta)^{r-1} \leq \prod_{i=1}^{r} V(P_i, \Delta[n-1])$$

thanks to a theorem by Bernstein, Kushnirenko and Khovanskii.

Let $f_1, ..., f_n : \mathbb{R}^n \to \mathbb{R}$ be polynomials.
Let $X = X_2 \cap ... \cap X_n$ of dimension 1, and $Y = X_1$ (codim.1) . Then Bezout inequality :

$$deg(X \cap Y) \leq deg(X)deg(Y) \qquad [B]$$

translates to

$$V_n(P_1, ..., P_n)V_n(\Delta) \leq V_n(P_2, ..., P_n, \Delta)V_n(P_1, \Delta[n-1]).$$

(recover previous inequality [B], by using [B] $r - 1$ times)

$$V_n(L_1, ..., L_n)V_n(\Delta) \leq V_n(L_2, ..., L_n, \Delta)V_n(L_1, \Delta[n-1]).$$

Since the inequality is invariant under replacing L_1 with $\lambda L_1 + x$, we may assume $L_1 \subset \Delta$, and $r(\Delta, L_1) = 1$, which implies $h_{L_1}(u_j) = h_\Delta(u_j)$ for all outer normals u_j, $j \leq n+1$, of Δ.

- In this case :

$$V(L_1, \Delta[n-1]) = \frac{1}{n}\sum_{j=1}^{n+1} h_{L_1}(u_j)Vol_{n-1}(K^{u_j}) = V_n(\Delta)$$

- therefore [B] follows from monotonicity of mixed volume.

- Let $K, L \in \mathcal{K}_n$. The inradius of K relative to L is $r(K, L) := \max\{\lambda > 0 : x + \lambda L \subset K, x \in \mathbb{R}^n\}$.
- A corollary of Diskant's inequality :

$$r(K, L)^{-1} \leq n \frac{V_1(K, L)}{Vol(K)} = n \frac{V(K[n-1], L)}{Vol(K)}$$

- Using this, J. Xiao has shown (2019) :

$$V(L_1, ..., L_n)V(K) \leq nV(L_2, ..., L_n, K)V(L_1, K[n-1])$$

for any convex bodies $L_1, ..., L_n$, and for any K.

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- $C = [0,1]^n$ shows that n is sharp.

(to see that $b(C) = n$, take $L_i = [0, e_i]$ and use projection formula).

Proof of Xiao's upper bound

- Let $K, L \in \mathcal{K}_n$. The inradius of K relative to L is $r(K, L) := \max\{\lambda > 0 : x + \lambda L \subset K, x \in \mathbb{R}^n\}$.
- Replace L_1 with $L' := r(K, L_1)L_1 + x \subset K$ (L' maximally contained).
- $r(K, L_1)V(L_1, ..., L_n) = V(L', L_2, ..., L_n) \leq V(K, L_2, ..., L_n)$ (monotonicity)
- therefore :

$$\begin{aligned} V(L_1, ..., L_n) \leq & \quad r(K, L_1)^{-1} \quad V(K, L_2, ..., L_n) \\ & \leq \frac{nV(K[n-1], L_1)}{V_n(K)} V(K, L_2, ..., L_n). \end{aligned}$$

We define :

$$b_2(K) = \max_{L_1, L_2} \frac{V(L_1, L_2, K[n-2])V(K)}{V(L_1, K[n-1])V(L_2, K[n-1])} \geq 1$$

And similarly

$$b(K) = \max_{L_1, \ldots, L_n} \frac{V(L_1, \ldots, L_n)V(K)}{V(L_2, \ldots, L_n, K)V(L_1, K[n-1])} \geq 1$$

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So that :

- $b_2(\Delta) = b(\Delta) = 1$ (by BKK theorem, or directly with MV)
- $\forall K, 1 \leq b_2(K) \leq b(K)$;
- by [Diskant, Xiao] : $\max_K b(K) \leq n$.
- $\forall K,\ b(TK) = b(K)$, for any (full-rank) affine T.

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$\max_K b_2(K) \geq 2$ can be seen with $K = O_n$ (the l_1-ball).

Who are the maximizers ?

- We've just seen that for all K, $b(K) \leq n$,

  with equality when $K = C = [0,1]^n$.
- (open) Are there other convex bodies such that $b(K) = n$?

Fact : $b_2(C) = \frac{n}{n-1}$. Hence $b_2(C) < 2$ (for $n \geq 3$).

Open question

does $\max_K b_2(K)$ remain bounded when $n \to \infty$?

$\max_K b_2(K) \geq 2$ can be seen with $K = O_n$ (the l_1-ball). Is there any better lower bound on $\max_K b_2(K)$?

Who are the minimizers ?

Question [Soprunov-Zvavitch 2015]

For which K, do we have $b_2(K) = 1$?

$\rightarrow$ recall $b_2(K) = \max_{L_1,L_2} \frac{V(L_1,L_2,K[n-2])V(K)}{V(L_1,K[n-1])V(L_2,K[n-1])}$.

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In words, $b_2(K)$ is the least constant $C > 0$ such that:

$$V(L_1, L_2, K[n-2])V(K) \leq C \;\; V(L_1, K[n-1])V(L_2, K[n-1])$$

holds for any $L_1, L_2 \in \mathcal{K}_n$.

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Question[Saroglou-Soprunov-Zvavitch 2018]

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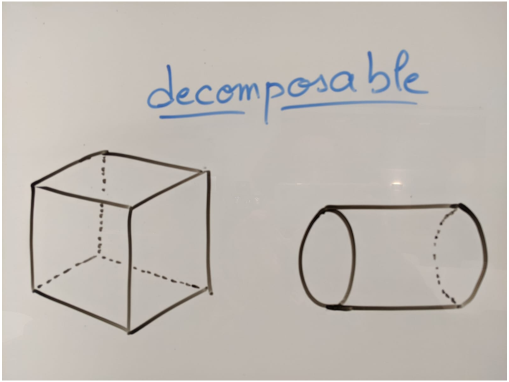
A standard definition

Definition

K is called decomposable if $\exists A, B \in \mathcal{K}_n$, $A \neq K$, such that $K = A + B$.



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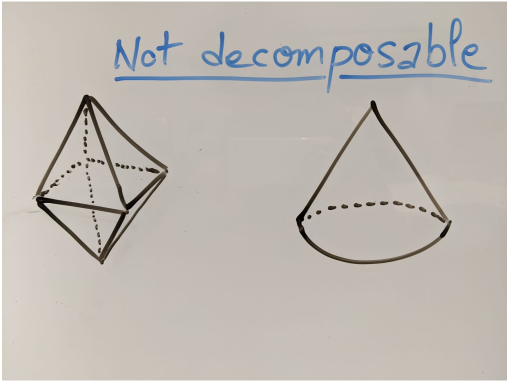

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Assume $K = A + B$, for some A, B.
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Equality implies A and B are homothetic.

Theorem[SSZ '18]

Let P be an n-polytope. If $b_2(P) = 1$, then $P = \Delta$.

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Assume $P = \bigcap_{i=1}^{N} H^-(u_i, h_i)$, i.e. (h_i) is the support vector of P.

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(tangential hyperplane to P^{u_i} moved by tu_i).

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This cannot be the case for all facets P^{u_i} of P, unless P is a simplex.

Characterization among all n-polytopes

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- In general it is not true that $S_{K+K_t} << S_K$.

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- if ∂K is somewhere locally smooth, then K is weakly decomposable. ($\rightarrow$ Wulff shape argument)

Open question : find a convex body K not weakly decomposable. ($K \neq \Delta$).

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Proposition [SZ'15], Prop 4.2 in [SSZ'18]

Assume there exists $u \in supp(S_K)$, such that K^u is 0-dimensional. Then $b_2(K) > 1$.

Some other known necessary conditions

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Theorem [SSZ'18]

Assume K is a convex body wih infinitely many facets. Then $b_2(K) > 1$.

A “dual” excluding condition

Let $K \in \mathcal{K}_n$, and $u \in \mathbb{S}^{n-1}$. Recall that $K^u = \{y \in K : \langle y, u \rangle = h_K(u)\}$. Denote $\Omega := supp(S_K) \subset \mathbb{S}^{n-1}$. Write $\Omega = \cup_{d=0}^{n-1} \Omega_d$, where $\Omega_d = \{u \in \Omega : K^u \text{ is } d\text{-dimensional}\}$.

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Theorem [S. 2022+]

Assume $S_K(\Omega_{n-2}) > 0$. Then $b_2(K) > 1$.

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Theorem [S. 2022+]

Assume $S_K(\Omega_{n-2}) > 0$. Then $b_2(K) > 1$.

Corollary

in $\mathbb{R}^3$, the simplex is the only minimizer of $b_2(K)$.

Proof (of corollary). Let $K \in \mathcal{K}_3$, write $supp(S_K) =: \Omega =: \Omega_0 \cup \Omega_1 \cup \Omega_2$. If $\Omega_0 \neq \emptyset$, then $b_2(K) > 1$, by [Prop. 4.2, SSZ'18]. Thus assume $\Omega = \Omega_1 \cup \Omega_2$.

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Let $K \in \mathcal{K}_n$, and $u \in \mathbb{S}^{n-1}$. Recall that $K^u = \{y \in K : \langle y, u \rangle = h_K(u)\}$. Denote $\Omega := supp(S_K) \subset \mathbb{S}^{n-1}$. Write $\Omega = \cup_{d=0}^{n-1} \Omega_d$, where $\Omega_d = \{u \in \Omega : K^u \text{ is } d\text{-dimensional}\}$.

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Assume $S_K(\Omega_{n-2}) > 0$. Then $b_2(K) > 1$.

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Note : this corollary was already known. It is proved in [SSZ'18].

An isoperimetric condition

Let $L \in \mathcal{K}_n$ be a k-dimensional. Denote :

$$Iso(L) := \frac{1}{k} \frac{Vol_{k-1}(\partial L)}{Vol_k(L)} =: \frac{1}{k} \frac{|\partial L|}{|L|}$$

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By the isoperimetric inequality :

$$Iso(L) = \frac{1}{d}\frac{|\partial L|}{|L|} = \frac{1}{d}\frac{|\partial L|}{|L|^{\frac{d-1}{d}}}\frac{1}{|L|^{1/d}} \geq \frac{1}{d}\frac{|\partial B_2^d|}{|B_2^d|^{\frac{d-1}{d}}}\frac{1}{|L|^{1/d}} = \frac{|B_2^d|^{1/d}}{|L|^{1/d}}.$$

thus if (F_k) is a sequence of facets with $Vol_{n-1}(F_k) \to 0$, then $Iso(F_k) \to +\infty$.

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Let $L \in \mathcal{K}_n$ be a k-dimensional convex body. Denote :

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Theorem [S. 2022+]

If $b_2(K) = 1$, then, for any affine transform T :

For any facet F of K : $Iso(TF) \leq Iso(TK)$.

(since $b_2(K)$ is affine invariant, while $\max_F \frac{Iso(F)}{Iso(K)}$, is not)

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- **Example** : the unit cube. It satisfies $Iso(C_n) = 2$, and so does any of its facets. Thus the criteria only allows to conclude $b_2(C_n) > 1$, after using an affine transform T.

Question : let $P \neq \Delta$. Does there necessarily exist T an affine transform, such that $\max_F Iso(TF) > Iso(TP)$?

Thank you !!