Multi-Bubble Isoperimetric Problems - Old and New

Emanuel Milman
Technion - Israel Institute of Technology
and Oden Institute at UT Austin

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joint work (in progress) with Joe Neeman (UT Austin)

The Classical Isoperimetric Inequality

"Among all sets in Euclidean space $\mathbb{R}^n$ having a given volume, Euclidean balls minimize surface area."

$$V(\Omega) = V(\mathrm{Ball}) \;\Rightarrow\; A(\Omega) \geq A(\mathrm{Ball}).$$

$\Omega \in \mathcal{B}(\mathbb{R}^n)$, $V = \mathrm{Leb}^n$, $A =$ Surface Area.

What is Surface Area? Various (non-equivalent) definitions:

- If $\partial\Omega$ smooth, $\int_{\partial\Omega} d\mathrm{Vol}_{\partial\Omega}$.
- Hausdorff measure $\mathcal{H}^{n-1}(\partial\Omega)$.
- Minkowski exterior boundary measure: $V^+(\Omega) = \liminf_{\epsilon\to 0+} \frac{V(\Omega_\epsilon \setminus \Omega)}{\epsilon}$, $\Omega_\epsilon := \{y \in \mathbb{R}^n \,;\, d(y,\Omega) < \epsilon\}$.
- De Giorgi Perimeter $P(\Omega) = \mathcal{H}^{n-1}(\partial^*\Omega) = \|1_\Omega\|_{BV} = \|\nabla 1_\Omega\|_{TV} = \sup\left\{\int_\Omega \nabla \cdot X \,;\, X \in C_c^\infty(\mathbb{R}^n; T\mathbb{R}^n)\,,\, |X| \leq 1\right\}$.
  Stronger than rest, l.s.c., invariant under null-set modifications.

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Isoperimetric Inequalities in Metric-Measure setting

Classical isoperimetric inequality is on $\mathbb{R}^n = (\mathbb{R}^n, |\cdot|, \text{Leb}^n)$.
Study in weighted-manifold setting $(M^n, g, \mu = \Psi(x) d\mathrm{Vol}_g)$, $\Psi > 0$.

Weighted Volume and Area:

- $V(\Omega) = \mu(\Omega) = \int_\Omega \Psi(x) d\mathrm{Vol}_g$.
- $A(\Omega) = P_\Psi(\Omega) = \int_{\partial^* \Omega} \Psi(x) d\mathcal{H}^{n-1}(x)$.

Denote $\mu^k = \Psi \mathcal{H}^k$, i.e. $\mu^{n-1} = \Psi \mathcal{H}^{n-1}$, $\mu^{n-2} = \Psi \mathcal{H}^{n-2}$, ...

Examples:

1. $\mathbb{S}^n = (\mathbb{S}^n, g_{\text{can}}, \lambda_{\mathbb{S}^n} = \frac{\mathrm{Vol}_{\mathbb{S}^n}}{\mathrm{Vol}(\mathbb{S}^n)})$ - P. Lévy, Schmidt 20-30's: geodesic balls are isoperimetric minimizers.
2. $\mathbb{G}^n = (\mathbb{R}^n, |\cdot|, \gamma^n = \frac{1}{(2\pi)^{n/2}} e^{-\frac{|x|^2}{2}} dx)$ - Sudakov–Tsirelson, Borell '75: half-spaces are isoperimetric minimizers.

Relation (Maxwell, Poincaré, Borel): $(\pi_{\mathbb{R}^n})_*(\lambda_{\sqrt{N}\mathbb{S}^N}) \to_{N \to \infty} \gamma^n$.

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Isoperimetric Inequalities for Clusters

Cluster $\Omega = (\Omega_1, \ldots, \Omega_q)$ is a partition $M = \Omega_1 \cup \ldots \cup \Omega_q$ (up to null-sets)
Given $V(\Omega) = (V(\Omega_1) \ldots V(\Omega_q))$ minimize $A(\Omega) = \frac{1}{2}\sum_{i=1}^q A(\Omega_i) = \sum_{i<j} A_{ij}$.

Previous examples: $q = 2$ $(\Omega_1 = U, \Omega_2 = M \setminus U)$, "Single Bubble".
Would like to study $q \geq 3$, "Multi Bubble" case.
Case $q = 3$ is called "Double Bubble" $(\Omega_1, \Omega_2, M \setminus (\Omega_1 \cup \Omega_2))$.

0. $\mathbb{R}^n$ - Theorem: for all $V(\Omega) = (v_1, v_2, \infty)$, standard double bubble (3 spherical caps meeting at $120°$ along $(n-2)$-dim sphere) minimizes total surface area:
   - $\mathbb{R}^2$ - F. Morgan's “SMALL" undergraduate group (Foisy–Alfaro–Brock–Hodges–Zimba) '93.
   - $\mathbb{R}^3$ - Hass–Hutchings–Schlafly '95 $v_1 = v_2$, Hutchings–Morgan–Ritoré–Ros '00.
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   - $\mathbb{R}^n$ - Reichardt '07.

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Isoperimetric Double-Bubble Conjectures

$q = 3$ regions in dimension $n \geq 2$:

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   $\mathbb{S}^2$ - Proved by Masters '96.
   $\mathbb{S}^3$ - Cotton–Freeman '02, Corneli–Hoffman-HLLMS '07, partial.
   $\mathbb{S}^n$ - Corneli–Corwin–Hoffman-HSADLVX '08, if $\left|v_i - \frac{1}{3}\right| \leq 0.04$.

2. $\mathbb{G}^n$ - Double-Bubble Conjecture: for all $V(\Omega) = (v_1, v_2, v_3)$, standard "tripod" / "Y" (3 half-hyperplanes meeting at $120°$ along $(n-2)$-dim plane) minimizes total (Gaussian) surface area.

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   Interaction between $\mathbb{G}$ and $\mathbb{S}$:
   $\mathbb{G}^2 \Rightarrow \mathbb{S}^N \;\; \forall N \gg 1 \Rightarrow \mathbb{S}^n \;\; \forall n \geq 2 \Rightarrow \mathbb{G}^n \;\; \forall n \geq 2$ by projection.

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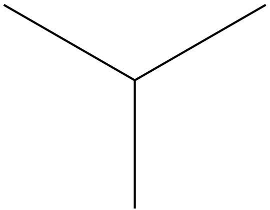
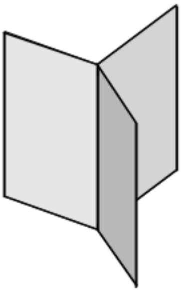
Y cone





Isoperimetric Multi-Bubble Conjectures

Higher-order cluster $\Omega = (\Omega_1, \ldots, \Omega_q)$.
There's no reasonable conjecture when $q \gg n$:

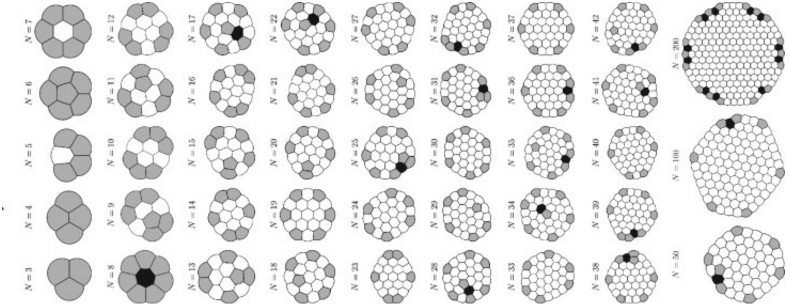


Image from Cox, Garner, et al.

Multi-Bubble Conjecture on $\mathbb{R}^n$ (J. Sullivan '95): If $q - 1 \leq n + 1$, for all $V(\Omega) = (v_1, \ldots, v_{q-1}, \infty)$, the minimizer is a standard $q-1$ bubble:

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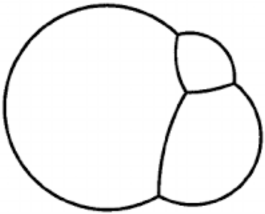

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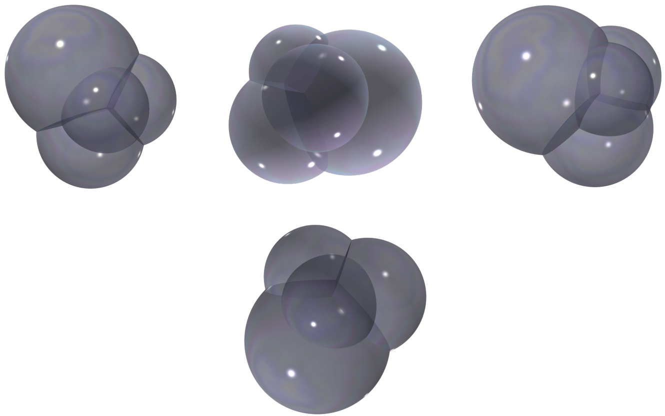

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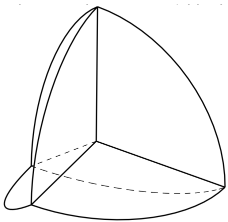

Montesinos Amilibia '01 - standard bubbles exist and are uniquely determined (up to isometries) for all prescribed volumes, for all $q-1 \leq n+1$.

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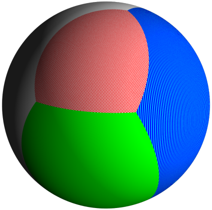

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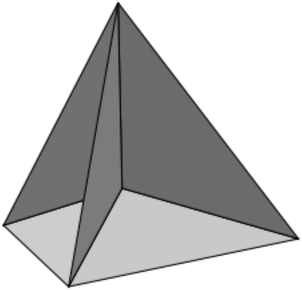

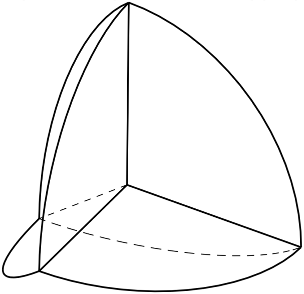

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$q = 2$ corresponds to the classical isoperimetric inqs.
$q = 3$ is the double-bubble theorem ($\mathbb{R}^n$) / conjecture ($\mathbb{S}^n$ / $\mathbb{G}^n$, $n \geq 3$).
$q = 4$ and $n = 2$ in $\mathbb{R}^n$ (planar triple-bubble) proved by Wichiramala '04.
Not aware of any other results when $q \geq 4$ prior to 2018.

Isoperimetric Multi-Bubble Results - Old

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Gaussian Double/Multi-Bubble Thm (M.–Neeman '18)

For all $n \geq 2$ and $2 \leq q \leq n+1$, the Multi-Bubble Conjecture on $\mathbb{G}^n$ is true: "a standard simplicial q-cluster is a Gaussian minimizer".

Gaussian Double/Multi-Bubble Uniqueness (M.–Neeman '18)

For all $n \geq 2$ and $2 \leq q \leq n+1$, simplicial q-clusters are the *unique* minimizers of Gaussian perimeter, up to null-sets.

In single-bubble setting ($q = 2$), uniqueness due to Ehrhard '86 and Carlen-Kerce '00.

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Multi-Bubble Conjecture on $\mathbb{S}^n$: If $q-1 \leq n+1$, for all $V(\Omega) = (v_1, \ldots, v_q)$, the minimizer is a standard $q-1$ bubble.

▸▸ Equal volume case?

1-2-3-4-5-Bubble Thm on $\mathbb{R}^n$ / $\mathbb{S}^n$ (M.–Neeman '22)

For all $n \geq 2$ and $2 \leq q \leq \min(6, n+1)$, the Multi-Bubble Conjecture on $\mathbb{R}^n$ / $\mathbb{S}^n$ is true: "A standard $q-1$ bubble is an isoperimetric minimizer". In other words, Double-Bubble ($n \geq 2$), Triple-Bubble ($n \geq 3$), Quadruple-Bubble ($n \geq 4$), Quintuple-Bubble ($n \geq 5$).

Multi-Bubble Uniqueness on $\mathbb{R}^n$ / $\mathbb{S}^n$ (M.–Neeman '22)

Uniqueness (up to null-sets) on $\mathbb{S}^n$ for $2 \leq q \leq \min(6, n+1)$.
Uniqueness (up to null-sets) on $\mathbb{R}^n$ for $2 \leq q \leq \min(5, n+1)$.

Q: Why is $\mathbb{S}^n$ case harder than $\mathbb{G}^n$? And $\mathbb{R}^n$ case even more so?
A1: $\mathbb{S}^N \Rightarrow \mathbb{G}^n$ by projection; $\mathbb{S}^n \Rightarrow \mathbb{R}^n$ by scale-invariance and shrinking to a point, but uniqueness is lost in both cases.

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1-2-3-4-5-Bubble Thm on $\mathbb{R}^n$ / $\mathbb{S}^n$ (M.–Neeman '22)

For all $n \geq 2$ and $2 \leq q \leq \min(6, n+1)$, the Multi-Bubble Conjecture on $\mathbb{R}^n$ / $\mathbb{S}^n$ is true: "A standard $q-1$ bubble is an isoperimetric minimizer". In other words, Double-Bubble ($n \geq 2$), Triple-Bubble ($n \geq 3$), Quadruple-Bubble ($n \geq 4$), Quintuple-Bubble ($n \geq 5$).

Multi-Bubble Uniqueness on $\mathbb{R}^n$ / $\mathbb{S}^n$ (M.–Neeman '22)

Uniqueness (up to null-sets) on $\mathbb{S}^n$ for $2 \leq q \leq \min(6, n+1)$.
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Single Bubble ($q = 2$):

0. $\mathbb{R}^n$ - symmetrization, Brunn–Minkowski, L^2, heat-flow, PDE, Localization, Optimal-Transport, Combinatorial, GMT.
1. $\mathbb{S}^n$ - symmetrization, GMT, Localization.
2. $\mathbb{G}^n$ - Projection of $\mathbb{S}^N$, symmetrization (Ehrhard), Brunn-Minkowski (Borell), Localization, heat-flow, GMT.

Double-Bubble ($q = 3$):

- Geometric Measure Theory (De Giorgi, Federer, Almgren, . . .) - existence and regularity of isoperimetric minimizers.
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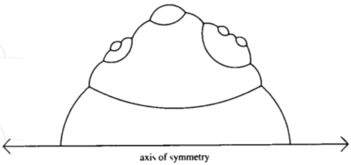

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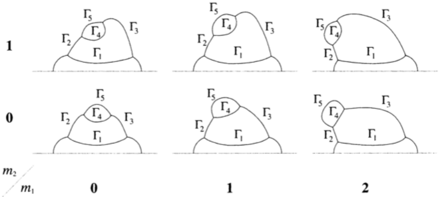

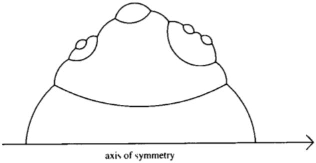

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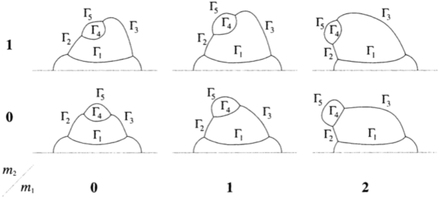

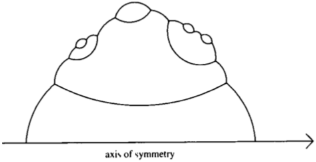

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Proof: Step 0 - Symmetry

Lemmas:

1. Simple symmetry on $\mathbb{R}^n/\mathbb{S}^n$: $\forall 2 \le q \le n+1$, **exists** minimizing q-cluster symmetric w.r.t. reflection about hyperplane H^{n-1}.

1b Full symmetry on $\mathbb{R}^n/\mathbb{S}^n$ (White, Hutchings '97): $\forall 2 \le q \le n$, **every** minimizing q-cluster is symmetric w.r.t. M^{q-2} ($M \in \{\mathbb{R}, \mathbb{S}\}$), i.e. invariant under all isometries which preserve every $x \in M^{q-2}$.

2. Product structure on $\mathbb{G}^n$ (M.–Neeman '18): $\forall 2 \le q \le n$, **every** stable (in particular, minimizing) q-cluster is a product $\tilde{\Omega} \times \mathbb{R}^{n+1-q}$.

Remarks:

- We don't need 1b in our approach.
- 1b and 2 reduce the problem to dimension $q-1$;
  1 does not reduce dimension.
- No expected symmetry / product structure in maximal case ($q = n+2$ in $\mathbb{R}^n/\mathbb{S}^n$, $q = n+1$ in $\mathbb{G}^n$) $\rightsquigarrow$
  Need separate argument for $\mathbb{G}^n$, out-of-reach on $\mathbb{R}^n/\mathbb{S}^n$.

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Proof on $\mathbb{S}^n$:

• Borsuk-Ulam Thm:
For any continuous $f : \mathbb{S}^n \to \mathbb{R}^n$ (or $\mathbb{R}^m, m \leq n$), $\exists \theta \in \mathbb{S}^n$ $f(\theta) = f(-\theta)$.

• Cor ("Ham-Sandwich"): $\exists H^{n-1} = \theta^\perp$ bisecting q-cells if $q \leq n+1$ (just use $f(\theta) = (2V(\Omega_i \cap \theta_+^\perp))_{i=1,\ldots,q-1} \in \mathbb{R}^{q-1}$).

• If Ω minimizer, $\Omega_\pm := \Omega \cap H_\pm^{n-1}$, reflect $\Omega_\pm$ about H^{n-1} – both have same volumes **and total perimeter** as Ω, otherwise one of $\Omega_\pm^{\text{sym}}$ would reduce it. □

• Remark $\partial_{\text{reg}}\Omega$ must meet bisecting H^{n-1} perpendicularly, otherwise could reduce perimeter of $\Omega_\pm^{\text{sym}}$ by smoothing the angle out.

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continue for $n+2-q$ steps.

Obtain minimizing cluster Ω^{sym} symmetric w.r.t. reflection in mutually perpendicular hyperplanes $\theta_1^\perp, \ldots, \theta_{n+2-q}^\perp$ ("unconditional").

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On smooth $(M^n, g, \mu^n = e^{-W} d\mathrm{vol})$, finite volume, GMT guarantees:

- Minimizing $\Omega = (\Omega_1, \ldots, \Omega_q)$ exists (Almgren: also on $\mathbb{R}^n$); cells are open, $\overline{\partial^* \Omega_i} = \partial \Omega_i$. Denote interfaces: $\Sigma_{ij} := \partial^* \Omega_i \cap \partial^* \Omega_j$.
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- Test against competitors by flowing along vector-field. If $X \in C_c^\infty(M^n; TM^n)$, $\frac{d}{dt} F_t = X \circ F_t$ diffeomorphism, $\Omega_t = F_t(\Omega)$. $V = V(\Omega_t)$, $A = A(\Omega_t)$, $\delta_X^k V = (\frac{d}{dt})^k|_{t=0} V(\Omega_t)$, $\delta_X^k A = (\frac{d}{dt})^k|_{t=0} A(\Omega_t)$.
- Since Ω (globally) minimizes area under volume constraint, there are Lagrange multipliers $\lambda \in E^{(q-1)} = \{ v \in \mathbb{R}^q \; ; \; \sum_{i=1}^q v_i = 0 \}$, s.t.:

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- Minimizing $\Omega = (\Omega_1, \ldots, \Omega_q)$ exists (Almgren: also on $\mathbb{R}^n$); cells are open, $\overline{\partial^* \Omega_i} = \partial \Omega_i$. Denote interfaces: $\Sigma_{ij} := \partial^* \Omega_i \cap \partial^* \Omega_j$.
- Almgren 70's: Σ_{ij} are C^∞ embedded mnflds w/ good properties. Great books on clusters by F. Morgan and F. Maggi.
- Test against competitors by flowing along vector-field. If $X \in C_c^\infty(M^n; TM^n)$, $\frac{d}{dt} F_t = X \circ F_t$ diffeomorphism, $\Omega_t = F_t(\Omega)$. $V = V(\Omega_t)$, $A = A(\Omega_t)$, $\delta_X^k V = (\frac{d}{dt})^k|_{t=0} V(\Omega_t)$, $\delta_X^k A = (\frac{d}{dt})^k|_{t=0} A(\Omega_t)$.
- Since Ω (globally) minimizes area under volume constraint, there are Lagrange multipliers $\lambda \in E^{(q-1)} = \{v \in \mathbb{R}^q \; ; \; \sum_{i=1}^q v_i = 0\}$, s.t.:

  Ω is "stationary" (critical point) $\quad \delta_X^1 A - \langle \lambda, \delta_X^1 V \rangle = 0.$

  Ω is "stable" (local minimizer) $\quad \delta_X^1 V = 0 \Rightarrow \delta_X^2 A - \langle \lambda, \delta_X^2 V \rangle \geq 0.$

- Since the first-variation of (weighted) area is (weighted) mean-curvature, then $H_{\Sigma_{ij},\mu} = \lambda_i - \lambda_j$ is constant (CMC) on Σ_{ij}.
- $\Sigma^1 := \cup_{i<j} \Sigma_{ij}$ has no boundary in weak sense ($\int_{\Sigma^1} d\omega^{n-2} = 0$). So if Σ_{ij}, Σ_{jk}, Σ_{ki} meet in threes, it must be in $120°$ angles.

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2 Product structure on $\mathbb{G}^n$ (M.–Neeman '18): $\forall 2 \leq q \leq n$, **every** stable (in particular, minimizing) q-cluster is a product $\tilde{\Omega} \times \mathbb{R}^{n+1-q}$.

Proof: Gaussian conjectured minimizers are generated by Translation group; its generators are $T_\theta \equiv \theta$ constant vector-fields.

Define:

- $\mathbb{R}^n \ni \theta \mapsto M\theta := \delta^1_{T_\theta} V = \left(\int_{\partial^* \Omega_i} \langle \theta, \mathfrak{n}_i \rangle \, d\gamma^{n-1}\right)_{i=1,\ldots,q} \in E^{(q-1)}$.
- $\mathcal{N} := \mathrm{span}(\mathfrak{n}|_{\Sigma^1})$; easy to show $\Omega = \tilde{\Omega} \times \mathcal{N}^\perp$, $\tilde{\Omega} \subset \mathcal{N}$.

Claim: $\mathcal{N}^\perp = \ker M$; would yield $\dim \mathcal{N}^\perp = \dim \ker M \geq n+1-q \rightsquigarrow \square$.

Proof: $\subseteq$ is trivial; $\supseteq$: let $\theta \in \ker M$, i.e. $\delta^1_{T_\theta} V = 0$. By stability:

$$0 \leq Q(T_\theta) =_{\text{calculation}} = -\int_{\Sigma^1} \langle \theta, \mathfrak{n} \rangle^2 \, d\gamma^{n-1} \leq 0 \;\; \Rightarrow \;\; \theta \perp \mathcal{N} \quad \square$$

Very lucky that $Q(T_\theta) \leq 0$! That's the difference with $\mathbb{R}^n / \mathbb{S}^n$, where conjectured minimizers are generated by Möbius group; $Q(W_\theta) \not\leq_? 0$.

Step 0: Product structure on $\mathbb{G}^n$

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Proof: Step 1 – Minimizer has Trivial Curvature

On $\mathbb{G}^n$: $q \leq n+1 \ \Rightarrow$ minimizer is flat $\mathrm{II} = 0$.
For $q < n+1$: use product structure $\Omega = \tilde{\Omega} \times \mathbb{R}^{n+1-q}$.
Maximal case $q = n+1$: separate argument, $Q(\text{Translations}) \leq 0$.

On $\mathbb{S}^n/\mathbb{R}^n$: $q \leq n+1 \ \Rightarrow$ minimizer is spherical $\mathrm{II}_0 = \mathrm{II} - \frac{H}{n-1}\mathrm{Id} = 0$.
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Higher codimension regularity

Regularity of higher codimension boundary (Morgan '94 $n=2$; Taylor '76 $n=2,3$; White '86, Colombo–Edelen–Spolaor '17 $n\geq 4$)

Let Ω be a minimizing q-cluster. Recall the cones $\mathbf{Y}\subset\mathbb{R}^2$, $\mathbf{T}\subset\mathbb{R}^3$.

1. $\Sigma := \cup_i \partial\Omega_i$ is the disjoint union of $\Sigma^1 := \cup_{i<j}\Sigma_{ij}, \Sigma^2, \Sigma^3, \Sigma^4$, where:
2. $\forall p\in\Sigma^2$ (triple pts), Σ is locally $C^{1,\alpha}$-diffeomorphic to $\mathbf{Y}\times\mathbb{R}^{n-2}$.
3. $\forall p\in\Sigma^3$ (quad pts), Σ is locally $C^{1,\alpha}$-diffeomorphic to $\mathbf{T}\times\mathbb{R}^{n-3}$.
4. Σ^4 (singular) is closed, $\mathcal{H}^{n-3}(\Sigma^4)=0$ (loc. finite $\mathcal{H}^{n-4}$-measure).

Hence $\Sigma^2=\cup_{i<j<k}\Sigma_{ijk}$. Denote $\partial\Sigma_{ij} := \cup_{k\neq i,j}\Sigma_{ijk}$; $(\Sigma_{ij},\partial\Sigma_{ij})$ incomplete.
By stationarity, $\forall p\in\Sigma_{ijk}$, $\Sigma_{ij},\Sigma_{jk},\Sigma_{ki}$ meet at 120° angles.

Kinderlehrer–Nirenberg–Spruck '78: in 2. regularity upgrades to C^∞.
Optimal regularity in 3. is open; $C^{1,\alpha}$ suspected to be optimal.

Local Integrability of Curvature (M.–Neeman '18)

For any compact K disjoint from Σ^4, $\mathrm{II}^{ij}\in L^2(\Sigma_{ij}\cap K), L^1(\partial\Sigma_{ij}\cap K)$.

Idea: using Schauder estimates, $\|\mathrm{II}^{ij}(p)\|\leq C_K/d(p,\Sigma^3)^{1-\alpha}$.

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Approximating scalar-fields - why and how?

Stability: $\delta^1_X V = 0 \;\Rightarrow\; 0 \leq Q(X) := \delta^2_X A - \langle \lambda, \delta^2_X V \rangle$ (the “index-form").

$$\delta^1_X V(\Omega_i) = \int_{\partial^* \Omega_i} X^{\mathfrak{n}_{ij}} d\mu^{n-1} = \sum_{j \neq i} \int_{\Sigma_{ij}} f_{ij} d\mu^{n-1} =: \delta^1_f V(\Omega_i),$$

where $f_{ij} = X^{\mathfrak{n}_{ij}} := \langle X, \mathfrak{n}_{ij} \rangle$ on Σ_{ij}. We'll call $f = (f_{ij})$ a “scalar-field".

Under favorable conditions: $Q(X) = Q_0(f)$, integration on $(\Sigma_{ij}, \partial \Sigma_{ij})$:

$$Q_0(f) = -\sum_{i<j} \left(\int_{\Sigma_{ij}} F(f, \nabla_\Sigma f, \|\mathrm{II}\|^2)\, d\mu^{n-1} + \int_{\partial \Sigma_{ij}} G(f, \mathrm{II}) d\mu^{n-2} \right).$$

Can justify if $\mathrm{supp}(X)$ disjoint from Σ^4; need integrability of II.

Stability for “physical" scalar-fields, $f_{ij} = X^{\mathfrak{n}_{ij}}$: $\delta^1_f V = 0 \Rightarrow Q_0(f) \geq 0$.

Working with scalar-fields is super convenient. Given smooth $f = (f_{ij})$ w/ $f_{ij} + f_{jk} + f_{ki} = 0$ on Σ_{ijk} (Kirchhoff), can we find smooth X w/ $X^{\mathfrak{n}_{ij}} = f_{ij}$?

Even for single-bubble, NO! Simons cone $\{x \in \mathbb{R}^8 : \sum_{i=1}^4 x_i^2 = \sum_{i=5}^8 x_i^2\}$.

So let's approximate: $\delta^1_X V = \delta^1_f V$ and $Q(X) \simeq Q_0(f)$.

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Approximating scalar-fields - why and how?

Stability: $\delta^1_X V = 0 \;\Rightarrow\; 0 \leq Q(X) := \delta^2_X A - \left\langle \lambda, \delta^2_X V \right\rangle$ (the “index-form").

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Formula for Index-Form

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$$Q_0(f) = \sum_{i<j} \left(- \int_{\Sigma_{ij}} f L_{Jac} f \, d\mu^{n-1} + \int_{\partial \Sigma_{ij}} f \left(\nabla_{\mathfrak{n}_{\partial ij}} f - \frac{\mathrm{II}^{ik}_{\partial\partial} + \mathrm{II}^{jk}_{\partial\partial}}{\sqrt{3}} f \right) d\mu^{n-2} \right).$$

L_{Jac} is the Jacobi operator:

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Problem: II a-priori unknown, no control over boundary's sign.

Idea 2.0: use stability for family of scalar-fields $f^a_{ij} = (a_i - a_j)\Psi$, $a \in \mathbb{R}^q$, so that $\delta^1_{f^a} V = 0$ and $\mathbb{E}_a Q_0(f^a) \leq 0$, $a \sim \mathbb{S}^{q-1}$. Read off information on II.

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No clear sign, not enough! Recall that standard-bubbles generated by Möbius group. Modding out isometries (& scaling), its generators are:

$$W_\theta := \begin{cases} \frac{|p|^2}{2}\theta - \langle\theta,p\rangle\,p & \text{on } \mathbb{R}^n \\ \theta - \langle\theta,p\rangle\,p & \text{on } \mathbb{S}^n \end{cases} \quad (\text{"}dilation-fields\text{"}).$$

These are conformal Killing-fields = generate 1-parameter family of conformal maps; ∇W_θ = Anti-Sym + f_pId ($f_p = 0$ for Killing). Properties:

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- On $\mathbb{S}^n$, by stability (applied twice!):

1. $\forall i,j \quad \int_{\Sigma_{ij}} \langle N,p\rangle \, dp = 0 \;\Rightarrow\; 0 \leq Q_0^{\mathrm{tr}}(\langle N,p\rangle)$.
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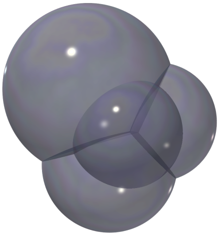
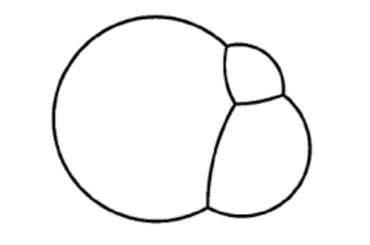
Is isotropicity obvious?





Proof: Steps 2 & 3 – Minimizer is Voronoi Cluster

On $\mathbb{G}^n$: These steps not needed; jump to Step 4!

On $\mathbb{S}^n/\mathbb{R}^n$: $q \leq n+1 \;\Rightarrow\;$ minimizer is spherical Voronoi cluster:
There exist $\{\mathfrak{c}_i\}_{i=1,\ldots,q} \subset \mathbb{R}^{n+1}/\mathbb{R}^n$ and $\{\kappa_i\}_{i=1,\ldots,q} \subset \mathbb{R}$ so that:

1. For every $\Sigma_{ij} \neq \emptyset$, Σ_{ij} lies on a (generalized) geodesic sphere S_{ij} with quasi-center $\mathfrak{c}_{ij} = \mathfrak{c}_i - \mathfrak{c}_j$ and curvature $\kappa_{ij} = \kappa_i - \kappa_j$.
   The quasi-center $\mathfrak{c} := \mathfrak{n} - \kappa p$ is constant on a sphere $S \subset \mathbb{S}^n/\mathbb{R}^n$.
2. On $\mathbb{S}^n$, the following Voronoi representation holds:

$$\Omega_i = \mathrm{int}\left\{p \in \mathbb{S}^n \; ; \; \arg\min_{j=1,\ldots,q} \langle \mathfrak{c}_j, p\rangle + \kappa_j = i\right\} = \bigcap_{j \neq i} \left\{p \in \mathbb{S}^n \; ; \; \langle \mathfrak{c}_{ij}, p\rangle + \kappa_{ij} < 0\right\}.$$

   Similarly on $\mathbb{R}^n$, after stereographic projection to $\mathbb{S}^n$.

Furthermore, each Ω_i is connected.

Step 2 involves simplicial homology of $\{\Omega_i\}_{i=1,\ldots,q}$, Convex Geometry.
Step 3 involves stability again, elliptic regularity, maximum principle.

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Furthermore, each Ω_i is connected.

Step 2 involves simplicial homology of $\{\Omega_i\}_{i=1,\ldots,q}$, Convex Geometry.
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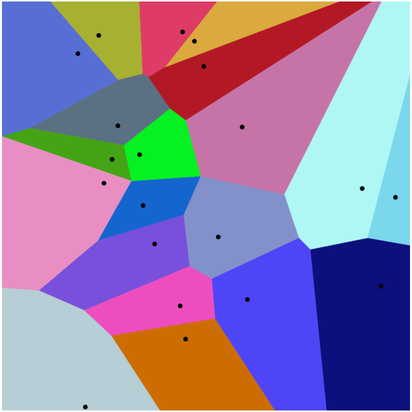

Euclidean Voronoi Cells:

$$\Omega_i = \{x : \arg\min_j |x - x_j|^2 = i\}$$

There exist $\{\mathfrak{c}_i\}_{i=1,\ldots,q} \subset \mathbb{R}^{n+1}/\mathbb{R}^n$ and $\{\kappa_i\}_{i=1,\ldots,q} \subset \mathbb{R}$ so that:

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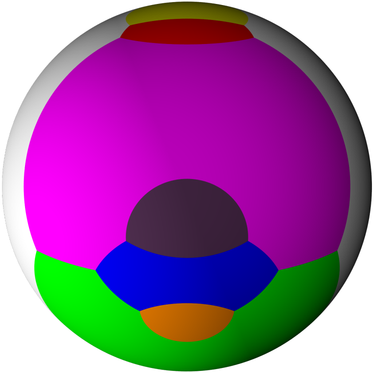

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An interlude – Lemma in Convex Geometry

From Almost Local to Global Convexity (M.–Neeman '18)

Let Ω be an open connected subset of $\mathbb{R}^n$, and let $B \subset \partial\Omega$ with $\mathcal{H}^{n-2}(B) = 0$. Assume that $\forall p \in \partial\Omega \setminus B$ there exists an open neighborhood N_p of p so that $\Omega \cap N_p$ is convex. Then Ω is convex.

- Classical for $B = \emptyset$ (Tietze, Nakajima 1928).
- False without connectedness, open / closed, $\mathcal{H}^{n-\alpha}$ for $\alpha < 2$.

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We are almost done! **Fact:** class of Voronoi clusters with $\Sigma_{ij} \neq \emptyset \; \forall i < j$ coincides with the class of conjectured minimizers.

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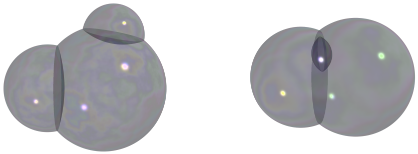
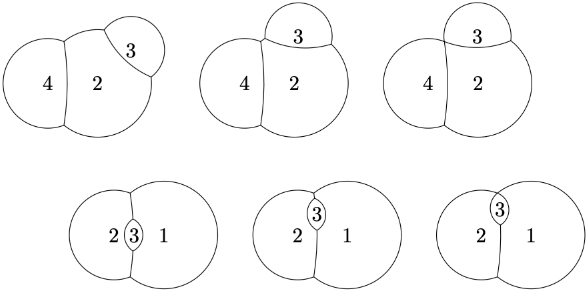
Double and Triple bubble on $\mathbb{R}^n/\mathbb{S}^n$

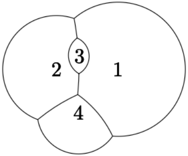




This already concludes proof of double/triple-bubble on $\mathbb{R}^n/\mathbb{S}^n$!

Quadruple bubble on $\mathbb{R}^n/\mathbb{S}^n$

For quadruple-bubble, analyze adjacency graphs on $q = 5$ vertices. Many graphs, but most are ruled out after showing that the minimal degree ≥ 3:



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For $q \gg 1$, leads to questions on incidence structure of $\{\Omega_i\}_{i=1,\ldots,q}$.
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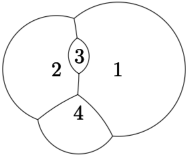

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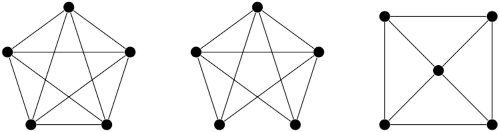

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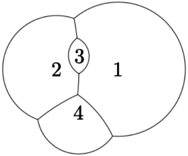

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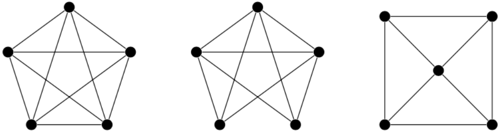

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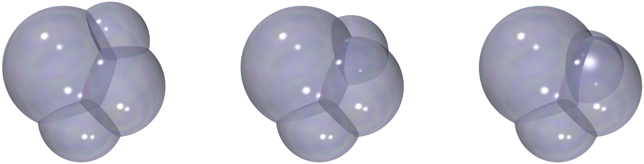
Ruling out $K_5 \setminus \{e\}$



The Isoperimetric Profile for Multi-Bubbles

$(M^n, g, \mu) \in \{\mathbb{G}^n, \mathbb{S}^n\}$. Need finite volume, so cannot work on $\mathbb{R}^n$.
$V(\Omega) = (V(\Omega_1), \ldots, V(\Omega_q)) \in \Delta^{(q-1)} := \left\{v \in \mathbb{R}^q \,;\, v_i \geq 0 \,,\ \sum_{i=1}^q v_i = 1\right\}$.

Isoperimetric Profile: $I^{(q-1)} : \Delta^{(q-1)} \to \mathbb{R}_+$,

$$I^{(q-1)}(v) := \inf \{A(\Omega); V(\Omega) = v\} .$$

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can show that this is well-defined; extend continuously to $\partial\Delta^{(q-1)}$.

Obviously $I^{(q-1)} \leq I_m^{(q-1)}$; want to show: $I^{(q-1)} \geq I_m^{(q-1)}$ on $\Delta^{(q-1)}$.
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Partial Differential Inequality for Profile

On $\mathbb{G}^n$, one can show that a fully non-linear elliptic PDE holds:

$$\mathrm{tr}((-\nabla^2 \mathcal{I}_m)^{-1}) = 2\mathcal{I}_m \text{ on } \Delta^{(q-1)}.$$

Similar (but more complicated) PDE holds on $\mathbb{S}^n$.

If we could show that the following PDI holds (in the viscosity sense):

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since $\mathcal{I} = \mathcal{I}_m$ on $\partial\Delta^{(q-1)}$ by induction, $\mathcal{I} \geq \mathcal{I}_m$ by maximum-principle.

This is our global information!! PDI takes into account entire $\Delta^{(q-1)}$.
Key idea: instead of using global information in space parameters $\mathbb{G}^n$, PDI propagates global information in volume parameters $\Delta^{(q-1)}$.

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Where does this PDE come from?
In the single-bubble case, $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-|x|^2/2}$, $\Phi(y) = \int_{-\infty}^{y} \varphi(x) dx$:

$$\mathcal{I}_{\mathbb{G}^n}(v) = \mathcal{I}_{\mathbb{G}^1}(v) = \{\varphi(a) \; ; \; \Phi(a) = v\} = \varphi \circ \Phi^{-1}(v).$$

Hence:

$$\mathcal{I}'(v) = \frac{\varphi'}{\varphi} \circ \Phi^{-1}(v) = -\Phi^{-1}(v) \; , \; \mathcal{I}''(v) = -\frac{1}{\varphi \circ \Phi^{-1}}(v) = -\frac{1}{\mathcal{I}(v)}.$$

Hence:

$$(-\mathcal{I}'')^{-1} = \mathcal{I} \text{ on } [0,1] \text{ (would be } 2\mathcal{I} \text{ on } \Delta^{(1)}).$$

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Upper bounding $\nabla^2\mathcal{I}$ via $Q(X)$

Recall $\frac{d}{dt}F_t = X \circ F_t$ diffeo, $\Omega_t = F_t(\Omega)$, $\mathcal{I}(V(\Omega_t)) \leq A(\Omega_t)$. Hence:

$$\left\langle \nabla\mathcal{I}, \delta^1_X V \right\rangle = \delta^1_X A = \left\langle \lambda, \delta^1_X V \right\rangle \Rightarrow \nabla\mathcal{I} = \lambda.$$

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This generalizes stability: $\delta^1_X V = 0 \;\Rightarrow\; 0 \leq Q(X)$.

The goal: choose X well to get a sharp PDI for $\mathcal{I}$.

Natural idea: use generators X of group generating conjectured minimizers (modulo isometries of the space)!

- $\mathbb{G}^n$ - Translation group generated by $T_\theta \equiv \theta$ constant fields.
- $\mathbb{S}^n$ - Möbius group generated by $W_\theta = \theta - \langle \theta, p \rangle p$ "dilation-fields".

This definitely yields sharp upper bounds on $\nabla^2\mathcal{I}$.

Problem: cannot a-priori exclude that cluster is lower-dimensional:

- $\mathbb{G}^n$ - $\Omega = \tilde{\Omega} \times \mathbb{R}^{n-d}$, $\tilde{\Omega}$ cluster on $\mathbb{R}^d$, $d < q-1$.
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In this case, the generators will only yield $d < q-1$ independent inqs, which is not enough to bound $\nabla^2\mathcal{I}$ on $E^{(q-1)} = T\Delta^{(q-1)}$.

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- $\mathbb{G}^n$ - Translation group generated by $T_\theta \equiv \theta$ constant fields.
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This definitely yields sharp upper bounds on $\nabla^2\mathcal{I}$.

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Upper bounding $\nabla^2 \mathcal{I}$ via $Q(X)$

Recall $\frac{d}{dt} F_t = X \circ F_t$ diffeo, $\Omega_t = F_t(\Omega)$, $\mathcal{I}(V(\Omega_t)) \leq A(\Omega_t)$. Hence:

$$\langle \nabla \mathcal{I}, \delta^1_X V \rangle = \delta^1_X A = \langle \lambda, \delta^1_X V \rangle \Rightarrow \nabla \mathcal{I} = \lambda.$$
$$(\delta^1_X V)^T \, \nabla^2 \mathcal{I} \, \delta^1_X V \leq \delta^2_X A - \langle \nabla \mathcal{I}, \delta^2_X V \rangle = \delta^2_X A - \langle \lambda, \delta^2_X V \rangle =: Q(X).$$

This generalizes stability: $\delta^1_X V = 0 \;\Rightarrow\; 0 \leq Q(X)$.

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On $\mathbb{S}^n$: fields yielding sharp PDI exist (non-trivial). But we don't have explicit formula, unless cluster is (pseudo)-conformally-flat ($\{\mathfrak{c}_i, \kappa_i\}$).
E.g.: • when cluster is full-dimenional, i.e. affine-rank$\{\mathfrak{c}_i\}_{i=1}^q = q - 1$;
• if all bubbles have a mutual common point.
In those cases, we obtain the sharp PDI for $\mathcal{I}$.

But what if the cluster is not pseudo-conformally-flat???
While this should never happen, we cannot a-priori exclude this.
Using Step 5 (= some tricks), we can go up to $q \leq 6$ on $\mathbb{S}^n$.

So why is $\mathbb{R}^n/\mathbb{S}^n$ harder than $\mathbb{G}^n$?

	$\mathbb{G}^n$	$\mathbb{S}^n/\mathbb{R}^n$
Group Generating Minimizers	Translations	Möbius Transformations (Liouville, $n \geq 3$: constitute all conformal automorphisms)
Effect on curvature II?	Invariant under translation	$\mathrm{II}' = a_p \mathrm{II} + b_p \mathrm{Id}$. Sphericity preserved, but curvature changes
Conjectured Minimizers	Flat	Conformally Flat (CF) on $\mathbb{S}^n$ (great spheres)
We can show	Flat	Spherical; However, showing CF requires finding conformal map, i.e. extra parameters

Thank you for your attention!

Equal Volume Case in $\mathbb{S}^n$ and $\mathbb{R}^n$

Equal Volume Multi-Bubble on $\mathbb{S}^n$ (M.–Neeman '18)

On $\mathbb{S}^n$, for any $q \leq n+2$, if $V(\Omega_1) = \ldots = V(\Omega_q) = \frac{1}{q}$ then the unique minimizer is a standard bubble.

Proof: immediate consequence from $\mathbb{G}^n$, since spherical and Gaussian volume/area coincide for centered cones on $\mathbb{S}^n \subset \mathbb{G}^{n+1}$, and the unique equal volumes minimizer on $\mathbb{G}^{n+1}$ for $q \leq (n+1)+1$ is the centered simplicial cluster (whose cells are centered cones).

Equal Volume Triple-Bubble on $\mathbb{R}^3$ (Lawlor '22)

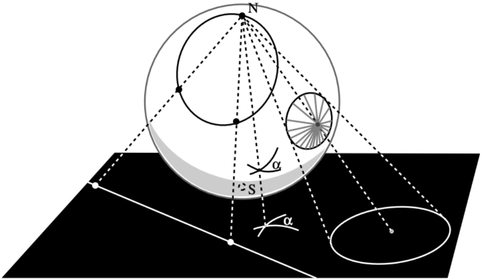
On $\mathbb{R}^3$, if $V(\Omega_1) = V(\Omega_2) = V(\Omega_3)$, then the unique (?) minimizer is a standard triple-bubble.

Jump back....

Möbius Group

Stereographic projection $T : \mathbb{S}^n \to \mathbb{R}^n$:

- T conformal = preserves angles $\langle dT\, u, dT\, v \rangle = c \langle u, v \rangle$.
- T preserves (generalized) spheres.



Stereographic projection preserves angles and takes circles to circles or lines

Taken from Delman–Galperin, “A tale of Three Circles".

What is the group generating standard bubbles?
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Thm (Liouville): All global conformal maps on $\mathbb{R}^n$ ($n \geq 3$) are Möbius.
Equivalent definitions of Möbius transformations on $\mathbb{R}^n$:

- Compositions of stereo-projections to and back $\mathbb{S}^n$;
- Compositions of spherical / hyperplane inversions;
- Compositions of isometries, scaling, and unit-sphere inversion.

(similarly on $\mathbb{S}^n$, by first stereographically projecting to $\mathbb{R}^n$).

So the Möbius group generates standard-bubbles on $\mathbb{R}^n/\mathbb{S}^n$.

Jump back...

Möbius Group

Stereographic projection $T : \mathbb{S}^n \to \mathbb{R}^n$:

- T conformal = preserves angles $\langle dT\, u, dT\, v\rangle = c \langle u, v\rangle$.
- T preserves (generalized) spheres.

What is the group generating standard bubbles?
(composition of stereographic projections is conformal map on $\mathbb{R}^n$).

Thm (Liouville): All global conformal maps on $\mathbb{R}^n$ ($n \geq 3$) are Möbius.
Equivalent definitions of Möbius transformations on $\mathbb{R}^n$:

- Compositions of stereo-projections to and back $\mathbb{S}^n$;
- Compositions of spherical / hyperplane inversions;
- Compositions of isometries, scaling, and unit-sphere inversion.

(similarly on $\mathbb{S}^n$, by first stereographically projecting to $\mathbb{R}^n$).

So the Möbius group generates standard-bubbles on $\mathbb{R}^n/\mathbb{S}^n$.

Jump back....