A Probabilistic Approach to Intersection Bodies

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Let $K \subset \mathbb{R}^n$ be a star body. The intersection body of K is defined by $\rho(IK, u) = |K \cap u^\perp|$.

Lutwak '88 , Fourier analytic method ⇝ Koldobsky

An isoperimetric inequality:

Thm (Busemann intersection): for $K \subseteq \mathbb{R}^n$ star body,

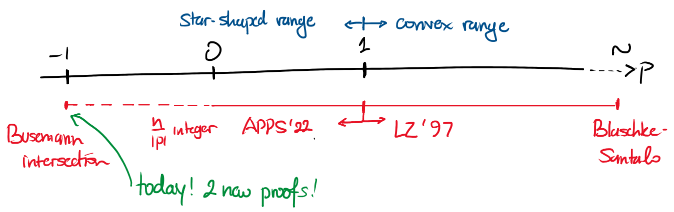
$$\int_{S^{n-1}} |K \cap \theta^\perp|^n \, d\theta \leq \frac{\kappa_{n-1}^n}{\kappa_n^{n-1}} |K|^{n-1}$$

$$|K \cap u^\perp| = \lim_{p \to -1} \frac{p+1}{2} \int_K |\langle x, u \rangle|^p \, dx$$

$p \geq 1$ polar p-centroid body $Z_p^\circ(K)$

Thm (Lutwak, Zhang '97): let K be a star body. For each $p \geq 1$, $|Z_p^\circ(K)| \leq |Z_p^\circ(K^*)|$ where $K^* = rB_2^n$ so that $|K^*| = |K|$.

A full spectrum of inequalities:



Lutwak '9? Gardner Giannopoulos '97
Haberl & Ludwig '06 '08 '09 Yaskin & Yaskina '06
Kalton Koldobsky Yaskin Yaskina '07

Dual Brunn-Minkowski Theory	Brunn-Minkowski theory
Intersection bodies	Explicitly convex construction: L_p centroid bodies
Desc: limit radial sums (Gardner, Weil '95)	Description: limit Minkowski sums
??	Isoperimetric inequality via symmetrization procedures: Lutwak & Zhang '97, Lutwak Yang Zhang '00, Campi & Gronchi '06
APPS'22: → random radial sum of convex bodies	Associate deterministic body with random objects, e.g. random p-centroid body (Paouris, Pivovarov '14, Cordero-Erausquin, Fradelizi, Paouris, Pivovarov '17)

II Define the empirical α-intersection body of a star body K by

$$\rho(I_N^\alpha K, u) = \frac{1}{N}\sum_{i=1}^{N} \left(|\langle X_i, u\rangle|^2 + \alpha^2 |u|^2 \right)^{-1/2}$$

where $X_1, \dots, X_N \sim \mathrm{Unif}(K)$ independent samples.

Where does this definition come from?

GW KKYY $\to$ intersection bodies

$=$ limit radial sum of ellipsoids

$$\text{RHS} = \oint \left(\text{support } \mathcal{E}^\alpha(X_i) = [-X_i, X_i] +_2 \alpha B_2^n\right)^{-1}$$

The convergence is given by the following:

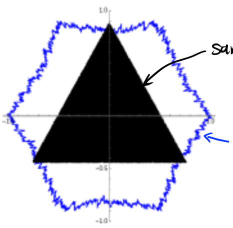
Lemma: $|IK| = \lim_{\alpha\to 0} s_\alpha^n \cdot \mathbb{E}|I_N^\alpha K|$ for some constant s_α.

A simulation:

$$\rho(I_N^\alpha K, u) = \frac{s_\alpha}{N}\sum_{i=1}^{N}\left(|\langle X_i, u\rangle|^2 + \alpha^2 |u|^2\right)^{-1/2} \qquad X_i \sim \mathrm{Unif}(K)$$

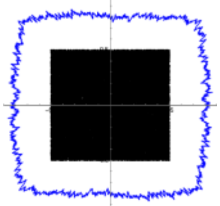
where s_α the normalization constant in the lemma.

With $N = 50000$ and $\alpha = 0.0001$, let's compute realizations of $I_N^\alpha K$ for various K on the plane. We normalize so that $|K| = 1$ and centroid of K is the origin.



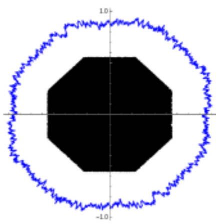
K_1 = equilateral triangle

$|I_N^\alpha K_1| \approx 2.417$



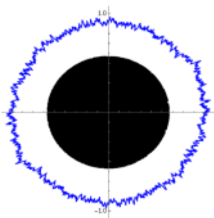
K_2 = square

$|I_N^\alpha K_2| \approx 2.523$



K_3 = regular octagon

$|I_N^\alpha K_3| \approx 2.538$



K_4 = unit disk

$|I_N^\alpha K_4| \approx 2.548$

An isoperimetric inequality for $I_N^\alpha K$

Thm (APPS '22): $\mathbb{E}|I_N^\alpha K| \le \mathbb{E}|I_N^\alpha K^*|$

Taking $\alpha \to 0$ and $N \to \infty$, we've proved the Busemann Intersection inequality.

For the next approximation method, we'll rewrite Busemann intersection inequality in a more convenient form.

Thm (Busemann intersection rewritten)

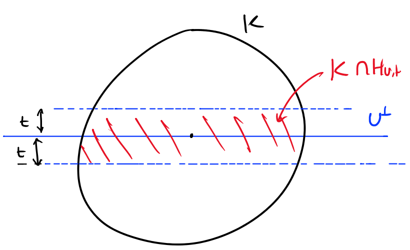
for $K \subseteq \mathbb{R}^n$ star body

$\mathbb{E}|K\cap\theta^\perp|^n \le \mathbb{E}|K^*\cap\theta^\perp|^n$, $\theta\sim \mathrm{Unif}(S^{n-1})$

II] Let $K \subseteq \mathbb{R}^n$ be an origin-symmetric convex body (for now)

Let $u\in\mathbb{R}^n$ and define $H_{u,t} = \{\, |\langle x,u\rangle| \le t \,\}$

Anttila, Ball, Perissinaki:



Estimating the slab's volume

Recall $|K\cap H_{u,t}| = \int_K \chi_{[-t,t]}(\langle x,u\rangle)\,dx$

Approximate empirically:

Define $\bar\rho_{N,t}(u) = \frac{1}{N}\sum_{i=1}^{N} \chi_{[-t,t]}(\langle X_i,u\rangle)$, $u\in\mathbb{R}^n$, $X_i\sim\mathrm{Unif}(K)$

Warning: this is not a true radial function, e.g. not homogenous in $|u|$

Another simulation

Let $N=5000$, $t=0.1$, and take 5000 Gaussian random points.

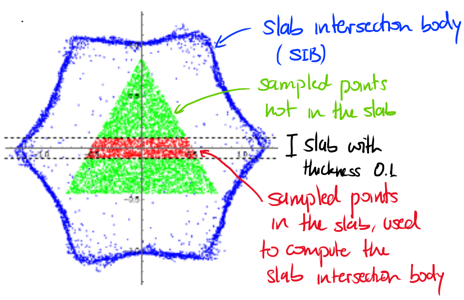
We'll compute

$$\tilde\rho(g_j) = \frac{1}{2(0.1)}\bar\rho_{5000,0.1}(g_j) = \frac{1}{5000}\sum_{i=1}^{5000}\chi_{[-0.1,0.1]}(\langle X_i,g_j\rangle)\cdot\frac{1}{2(0.1)}$$

for $X_i\sim\mathrm{Unif}(K)$ and $j=1,\dots,5000$. then plotting $\tilde\rho(g_j)\cdot g_j$

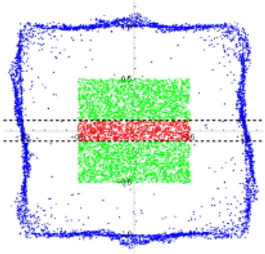
We normalize so that K has area 1 and centroid at $(0,0)$.

Area computed by averaging $\frac{1}{2}\tilde\rho^2(g_j)$



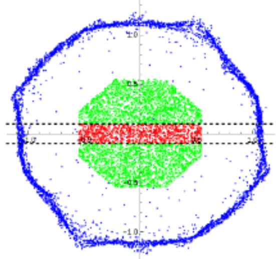
K_1 = equilateral triangle

$|\text{SIB for } K_1| \approx 1{,}202$



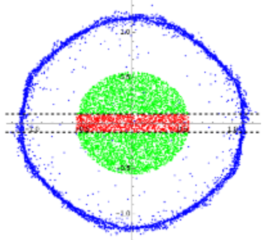
K_2 = square

$|\text{SIB of } K_2| \approx 1.254$



K_3 = octagon

$|\text{SIB of } K_3| \approx 1.203$



K_4 = unit disk

$|\text{SIB of } K_4| \approx 1.262$

Isoperimetric inequality for slab intersection bodies:

Thm (Pivovarov-S., 23+): Let $K \subseteq \mathbb{R}^n$ be star body.

For each $t > 0$ and $N \geq n$,

$$\mathbb{E}\, \bar{\rho}_{N,t}^{\,n}(g) \leq \mathbb{E}(\bar{\rho}_{N,t}^{*})^n(g)$$

where $g \sim N(0, I_n)$. and $\bar{\rho}_{N,t}^{*}$ is sampled from K^*.

Limiting argument will again give the Busemann intersection inequality.

A word about the proof of the isoperimetric-type theorems:

The key in both theorems is to write the wanted quantity set (e.g. empirical α-intersection bodies and slab intersection bodies) as a special average of convex objects and use results from the convex case.

The inequalities are given by the following symmetrization result:

Thm (Cordero-Erasquin, Fradelizi, Paouris, Pivovarov '15):

Let $X_1, \ldots, X_N \sim f$, f a probability density

$X_1^*, \ldots, X_N^* \sim f^*$, f^* symmetric decreasing rearrangement of f

C, origin-symmetric convex body

For a measure μ with radial decreasing density,

$$\mathbb{E}\, \mu((\underbrace{[X_1 \cdots X_N]}_{\mathbb{X}} C)^\circ) \leq \mathbb{E}\, \mu((\underbrace{[X_1^* \cdots X_N^*]}_{\mathbb{X}^\#} C)^\circ)$$

We'll sketch out the proof from each description. In what follows,

$X_i \sim \mathrm{Unif}(K)$ $\quad X_i^* \sim \mathrm{Unif}(K^*)$ $\quad C = \ell^1$ ball. $\quad \mu$ Gaussian measure

Recall

$$\rho(I_N^\alpha(K), u) = \frac{1}{N}\sum_{i=1}^{N} \underbrace{(|\langle X_i, u\rangle|^2 + \alpha^2 |u|^2)^{-1/2}}_{\text{reciprocal of support function of ellipsoid } \mathcal{E}^\alpha(X_i)}$$

To understand how this expression connects with the theorem above, let's pretend first that $\alpha = 0$. Then we have an expression of the form

$$\frac{1}{N}\sum_{i=1}^{N} |\langle X_i, u\rangle|^{-1}$$

But $|\langle X_i, u\rangle|^{-1} = h^{-1}([-X_i, X_i], u) \equiv h^{-1}(\mathbb{X}[-e_i, e_i], u) = h^{-1}([-e_i, e_i], \mathbb{X}^T u)$

and thus

$$\rho^n(I_N(K), u) = \frac{1}{N^n}\sum_{m=1}^{n}\sum_{n_1+\cdots+n_m=n}\binom{n}{n_1 \cdots n_m}\sum_{I=(i_1,\ldots,i_m)}\prod_{j=1}^{m} h^{-n_j}([-e_j, e_j], \mathbb{X}_{i_j}^T u)$$

But

$$\prod_{j=1}^{\ell} h^{-n_j}([-e_i, e_i], \mathbb{X}^T u) = \int_{(0,\infty)^\ell}\prod_{j=1}^{\ell}\Big[u \in \overbrace{\{h^{-1}([-e_j, e_j], \mathbb{X}_{i_j}^T \cdot) \geq t_j\}}^{\text{indicator of this set}}\Big]\, dt$$

$$= \int_{(0,\infty)^\ell}\Big[u \in \bigcap_{j=1}^{\ell}\{\underbrace{h(t_j^{-1}\mathbb{X}_{i_j}[-e_j, e_j], \cdot) \leq 1}\}\Big]\, dt$$

Key observation: this is a support function of the segment, i.e. the slab $\{|\langle X_i, \cdot\rangle|$, so what we have is an intersection of slabs, which polar is the (image of a) ℓ^1 ball.

$$= \int_{(0,\infty)^\ell} \left[u \in \left(\mathbb{X}_I \left(\underset{\text{diagonal of } t_i\text{'s}}{T} B_1^n \right) \right)^\circ \right] dF$$

where $\mathbb{X}_I = [X_{i_1} \cdots X_{i_\ell}]$

Replace u with a Gaussian vector g and take expectation to get

$$\underset{\text{in } g}{\mathbb{E}} \rho^n(I_N K, g) = \sum \cdots \int_{(0,\infty)^\ell} \gamma_n\Big(\underbrace{\left(\mathbb{X}_I (T B_1^n) \right)^\circ}_{\text{Apply CEFPP here: } \mathbb{X} \equiv \mathbb{X}_I \text{ and } C = TB_1^n.} \Big)\, dF$$

Note: Same principle applies to the proof for the slab intersection body.

Recall $\overline{\rho}_{N,t}(u) = \frac{1}{N} \sum_{i=1}^{N} \chi_{[-t,t]}(\langle X_i, u\rangle)$, $X_i \sim \mathrm{Unif}(K)$

Again we take the n-th power and expand. A typical term in the sum looks like

$$\prod_{i=1}^{m} \chi_{[-t,t]}(\langle X_i, u\rangle) = \{u \mid |\langle X_i, u\rangle| \le t \text{ for } i = 1, \dots, m\} \rightarrow \text{intersection of slabs}$$

$$= \left(\operatorname{conv}\{\pm t^{-1} X_i\}_{i=1}^{m} \right)^\circ$$

$$= \left(\mathbb{X}\, t^{-1} B_1^m \right)^\circ$$

Collecting all such terms gives us

$$\underset{\text{w.r.t. } g}{\mathbb{E}} \overline{\rho}_{N,t}^{\,n}(g) = \frac{1}{N^n} \sum_{m=1}^{n} m^n \sum_{\substack{I = (i_1, \dots, i_m) \\ \text{distinct}}} \gamma_n\Big(\underbrace{\left(\mathbb{X}_I t^{-1} B_1^m \right)^\circ}_{\text{apply CEFPP here}} \Big) \qquad \mathbb{X}_I = [X_{i_1} \cdots X_{i_m}]$$

The argument above is a special case of a more general theorem

Define the dual p-centroid body $Z^\diamond_{p,(C_i)}(K)$

$$\rho^{-p}(Z^\diamond_{p,C}(K), u) = \frac{1}{N} \sum_{i=1}^{N} h^p(C_i, \mathbb{X}_i u)$$

where $\mathbb{X}_i = [X_{i_1} \cdots X_{i_n}]$ (note the increased number of samples)

Then the radial function of $I_N^\alpha(K)$ can be seen as a special case of $Z^\diamond_{p,(C_i)}(K)$'s with $p = -1$ and certain choice of C_i

We can write $Z^\diamond_{-1,(C_i)}(K)$ as a pre-image of

$$B^N_{-1}(C_i) := \left\{ \sum_{i=1}^{N} h^{-1}(C_i, \cdot) \le 1 \right\}$$

by the matrix $\mathbb{X}^T = [\mathbb{X}_1 \cdots \mathbb{X}_N]^T$

Extra: This is one point of view that write $Z^\diamond_{p,C}(K)$ as a random sum. In the positive range, we used another: that these random star bodies are random sections of ℓ^p balls.

To be precise, on the level of volume,

$$|Z^\diamond_{p,C}(K)| = |\mathbb{X}^{-T} \underset{\text{analogous definition}}{B_p^N}(C_i)|$$

In particular if $C_i = [-e_i, e_i]$, we can express the slices as a Gaussian mixture with p-stable random variables using a formula of Nayar & Tkocz '20:

$$|Z^\diamond_{p,C}(K)| = C_{N,n,p}\, \mathbb{E}_W \underbrace{\sqrt{W_1 \cdots W_N}}_{p\text{-stable random variables}} \left| \left(\mathbb{X} (W \underset{\text{usual Euclidean ball}}{B_2^N}) \right)^\circ \right|$$