The Vector Balancing Constant for Zonotopes

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Joint work with Rainie Heck and Thomas Rothvoss

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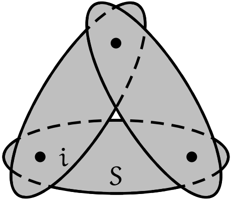
Discrepancy theory

- Set system $\mathcal{S} = \{S_1, \ldots, S_m\}, S_i \subseteq [n]$



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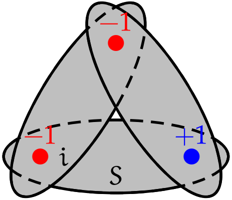

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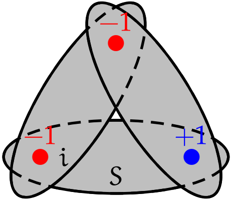

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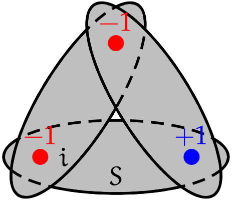

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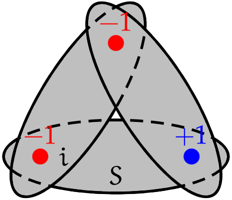

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- **Linear algebraic version:** For $A \in [-1,1]^{m \times n}$ there is a $x \in \{-1,1\}^n$ with $\|Ax\|_\infty \leqslant O(\sqrt{n \log \frac{2m}{n}})$.

The vector balancing constant

- For symmetric convex bodies $K, Q \subseteq \mathbb{R}^d$,

$$\mathrm{vb}(K, Q) := \sup \left\{ \min_{x \in \{-1,1\}^n} \Big\| \sum_{i=1}^{n} x_i v_i \Big\|_Q \mid n \in \mathbb{N}, v_1, \ldots, v_n \in K \right\}$$

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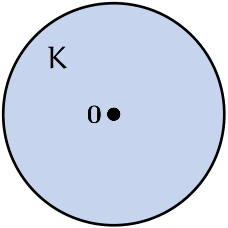

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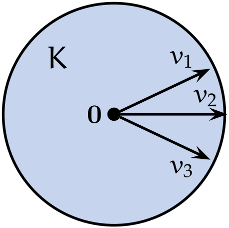

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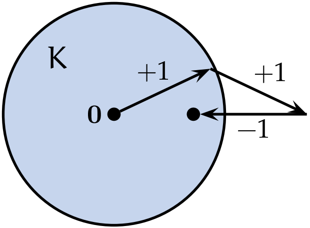

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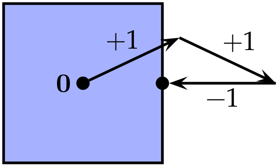


$\mathrm{vb}(K,Q) \cdot Q$

The vector balancing constant

- For symmetric convex bodies $K, Q \subseteq \mathbb{R}^d$,

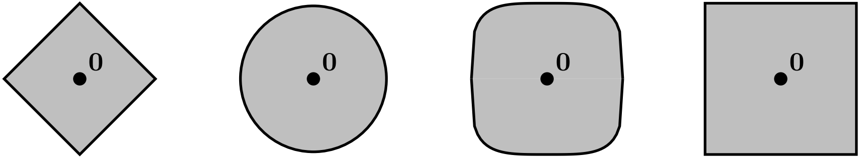
$$vb_n(K,Q) := \sup\left\{ \min_{x\in\{-1,1\}^n} \left\| \sum_{i=1}^{n} x_i v_i \right\|_Q \mid v_1,\dots,v_n \in K \right\}$$

Theorem (LSV'86)

One has $vb(K,Q) \leqslant 2 \cdot vb_d(K,Q)$.

The L_p-balls

- Let $B_p^d := \{x \in \mathbb{R}^d \mid \|x\|_p \leqslant 1\}$



B_1^d B_2^d B_4^d B_∞^d

The vector balancing constant (2)

Same bodies:

- **Spencer's Theorem.** $\mathrm{vb}(B_\infty^d, B_\infty^d) \lesssim \sqrt{d}$ and $\mathrm{vb}_n(B_\infty^d, B_\infty^d) \lesssim \sqrt{n \log(\frac{2d}{n})}$ for $n \leqslant d$.

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Different bodies:

- $vb(B_1^d, B_\infty^d) \leqslant 2$ [Beck, Fiala '81]
- **Komlós Conjecture:** $vb(B_2^d, B_\infty^d) \leqslant O(1)$
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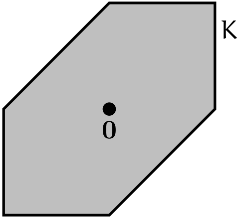
- (R. Rothvoss '20) $vb(B_4^d, B_\infty^d) \leqslant O(d^{1/4})$.

Zonotopes

Definition

A **zonotope** $K \subseteq \mathbb{R}^d$ is the projection of a cube.



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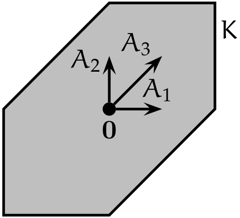
- We write $K = \{\sum_{i=1}^{m} y_i A_i \mid y \in [-1,1]^m\}$ where $A \in \mathbb{R}^{m \times d}$



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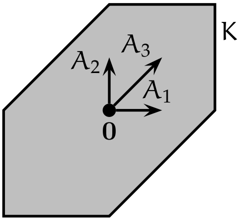

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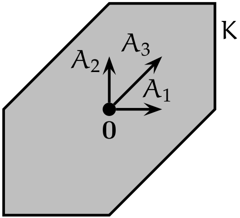

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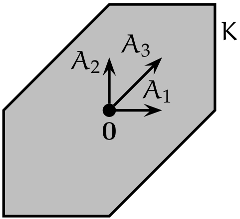

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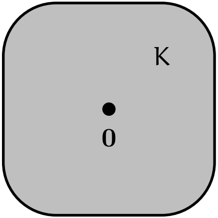
- NOT a zonoid: B_1^d

Reducing number of segments

Theorem (Talagrand '90)

For any zonotope $K \subseteq \mathbb{R}^d$ *there is a zonotope* Q *with* $O(\frac{d \log d}{\varepsilon^2})$ *segments so that* $Q \subseteq K \subseteq (1+\varepsilon)Q$.



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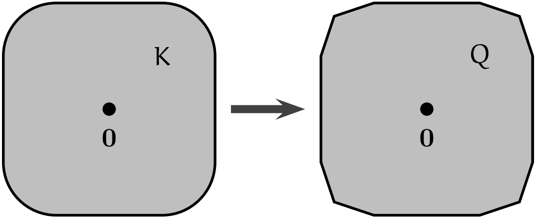

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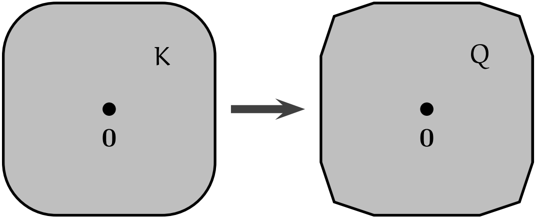

Question (Bourgain, Lindenstrauss, Milman '89)

Are $O_\varepsilon(d)$ segments enough?

vb of zonotopes – a warmup

Lemma (Folklore)

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- Then
  $\mathrm{vb}_d(K, K) \leqslant \mathrm{vb}_d(B_\infty^m, B_\infty^m) \leqslant O(\sqrt{d \log \frac{2m}{d}}) \leqslant O(\sqrt{d \log \log d})$. □

Our main contribution

(Schechtman 2002; Open problems on embeddings of finite metric spaces)

Is it true that for each zonotope $K \subseteq \mathbb{R}^d$ one has $\mathrm{vb}(K, K) \leqslant O(\sqrt{d})$?

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Theorem (Heck, R., Rothvoss 2022)

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Normalizing a zonotope

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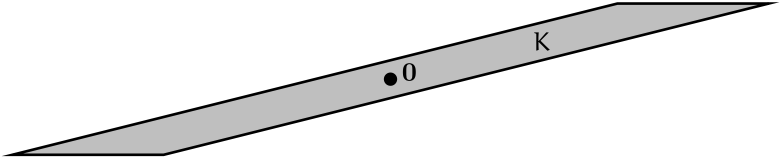
We call a zonotope $K \subseteq \mathbb{R}^d$ **normalized** if $K = \sqrt{\frac{d}{m}} A^T B_\infty^m$ where $A \in \mathbb{R}^{m \times d}$ has

- Orthonormal columns
- Short rows: $\|A_i\|_2 \leqslant 2\sqrt{\frac{d}{m}}$ for all $i \in [m]$



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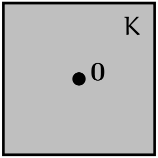

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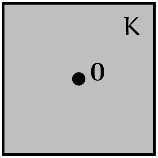
- Any zonotope can be made apx. normalized by a linear transformation + subdivision of segments (similar to [BLM' 89, Talagrand 90])
- B_∞^d is normalized



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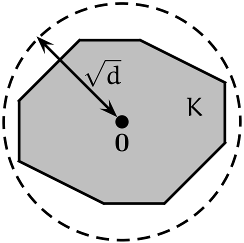

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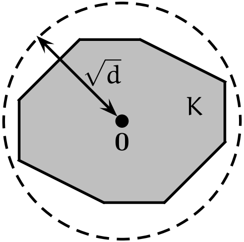

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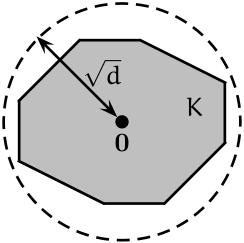

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$$\left\| \sqrt{\frac{d}{m}} A^T y \right\|_2 \leqslant \sqrt{\frac{d}{m}} \cdot \underbrace{\|A^T\|_{op}}_{\leqslant 1} \cdot \underbrace{\|y\|_2}_{\leqslant \sqrt{m}} \leqslant \sqrt{d}$$

Partial colorings

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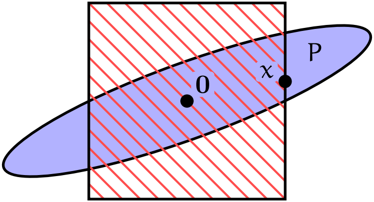
- Picture for case $v_i = e_i$:

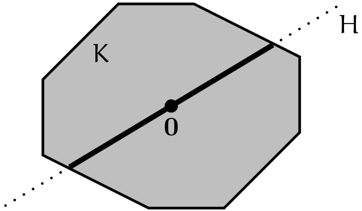
$[-1,1]^n$



Main technical contribution

Theorem (Heck, R., Rothvoss 2022)

For any normalized zonotope $K \subseteq \mathbb{R}^d$ and any n-dimensional subspace $H \subseteq \mathbb{R}^d$ one has $\gamma_H(K \cap H) \geqslant e^{-\Theta(n)}$.



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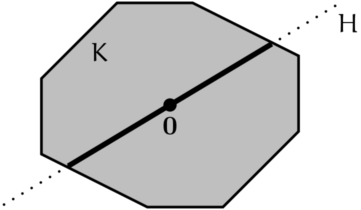

Corollary

For any $v_1, \ldots, v_n \in K$ there is a good partial coloring x with $\sum_{i=1}^n x_i v_i \in O(\sqrt{d}) \cdot K$.

- **Proof.** Use $\|v_i\|_2 \leqslant \sqrt{d}$. Then use partial col. lemma with $H := \text{span}\{v_1, \ldots, v_n\}$. $\square$

A first weak bound

Lemma

Let $K = A^{\mathsf{T}} B_\infty^m$ *where* A *has orthonormal columns and let* $H \subseteq \mathbb{R}^d$ *with* $n = \dim(H)$. *Then* $\gamma_H(K \cap H) \geqslant e^{-n}$.

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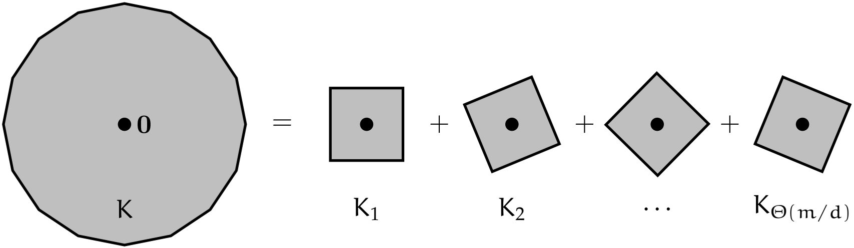
Decomposing a zonotope

Lemma

Any normalized zonotope K *can be written as the Minkowski sum of* $\Theta(\frac{m}{d})$ *zonotopes* K_j *so that* $\Theta(\frac{m}{d}) \cdot K_j$ *is approx. normalized*.*



* of the form $\tilde{A}^T B_\infty^{\tilde{m}}$ with $\sum_i \tilde{A}_i \tilde{A}_i^T \succeq \Omega(1) I_d$.

Decomposing a zonotope (2)

Theorem (Marcus, Spielman, Srivastava 2015)

Let $v_1, \ldots, v_m \in \mathbb{R}^d$ *so that* $\sum_{i=1}^m v_i v_i^T = I_d$ *and* $\|v_i\|_2^2 \leqslant \varepsilon$ *for all* $i \in [m]$. *There is a partition* $[m] = S_1 \dot{\cup} S_2$ *so that for both* $j \in \{1, 2\}$ *one has*

$$\left(\frac{1}{2} - 3\sqrt{\varepsilon}\right) I_d \preceq \sum_{i \in S_j} v_i v_i^T \preceq \left(\frac{1}{2} + 3\sqrt{\varepsilon}\right) I_d$$

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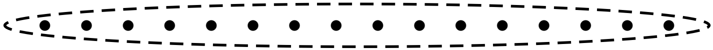

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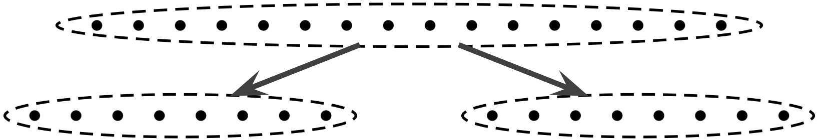

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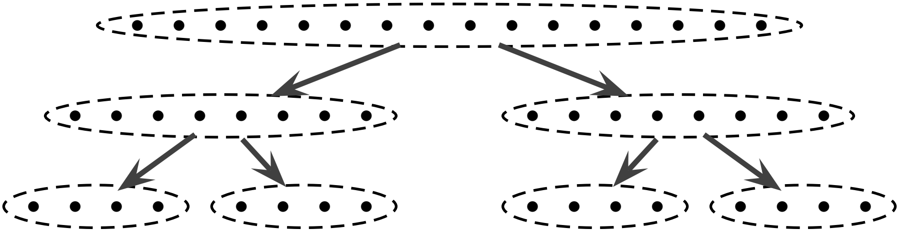

Using log-concavity

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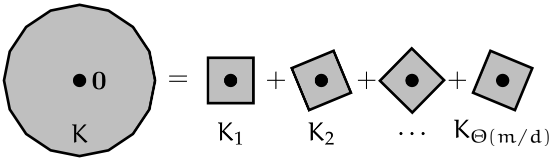

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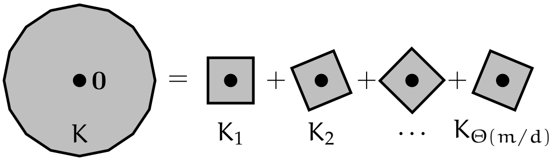

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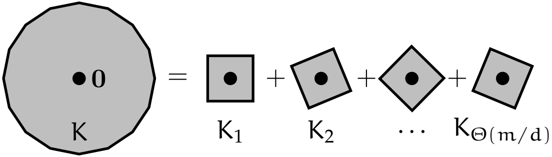

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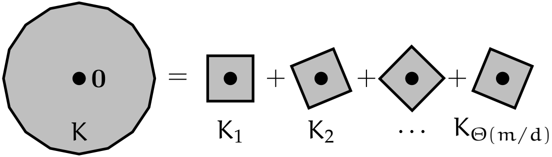

Finishing the proof the main Theorem

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setting $t := \log \log \log d$ and using $m \lesssim d \log d$. $\square$

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- True for $\|\cdot\|_2$, w. B consists of scaled rows of A [BSS'08]

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Thanks for your attention!