Brunn-Minkowski inequalities for path spaces on Riemannian surfaces

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The Brunn Minkowski inequality

Definition (Minkowski sum)

$A, B \subseteq \mathbb{R}^n$

$$A + B := \{a + b \mid a \in A, b \in B\},$$

$$\lambda A := \{\lambda a \mid a \in A\}.$$

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Theorem (Brunn-Minkowski)

$A, B \subseteq \mathbb{R}^n$ *nonempty Borel sets,* $0 < \lambda < 1$.

$$\mathrm{Vol}_n((1-\lambda)A + \lambda B)^{1/n} \geq (1-\lambda) \cdot \mathrm{Vol}_n(A)^{1/n} + \lambda \cdot \mathrm{Vol}_n(B)^{1/n}.$$

The Brunn Minkowski inequality - Riemannian setting

Definition (Riemannian Minkowski average)

(M, g) *Riemannian Manifold,* $\dim M = n$, $A, B \subseteq M$, $0 < \lambda < 1$,

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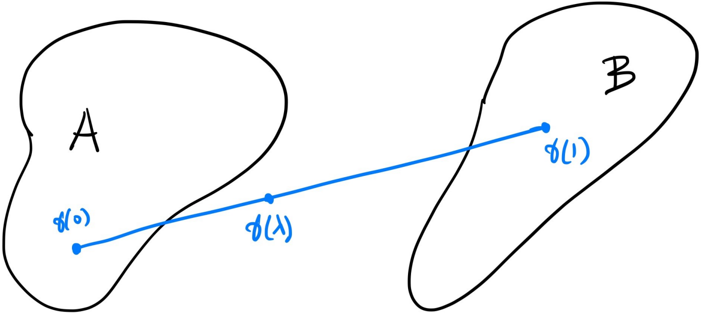
e.g. $\lambda = 1/2$*: all midpoints of minimizing geodesics joining* A, B*.*



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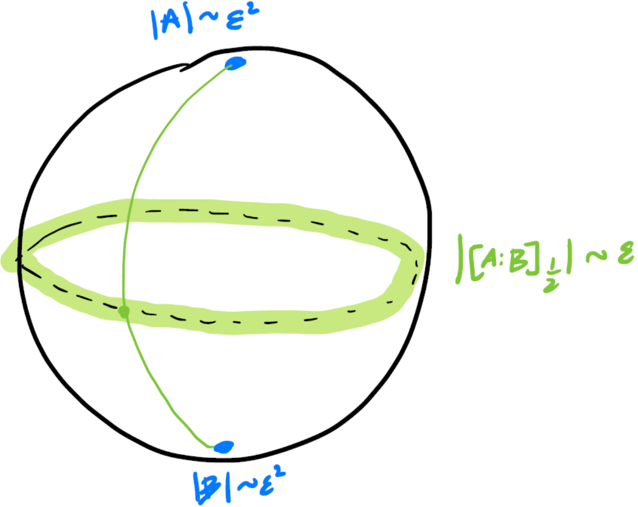
Theorem (Cordero-Erausquin, McCann, Schmuckenschläeger '01, Sturm '06)

(M, g) *complete Riemannian Manifold,* $\mathrm{Ric}_g \geq 0$,
$A, B \subseteq M$ *Borel, nonempty,* $0 < \lambda < 1$, $\implies$

$$\mathrm{Vol}_g([A : B]_\lambda)^{1/n} \geq (1 - \lambda) \cdot \mathrm{Vol}_g(A)^{1/n} + \lambda \cdot \mathrm{Vol}_g(B)^{1/n}.$$

Brunn - Minkowski on the Sphere



Brunn-Minkowski on the hyperbolic plane

$p, q \in \mathbf{H}$, $d(p,q) = \ell$, $\qquad A := B_1(p), B := B_1(q).$

$$\text{Area}\left([A:B]_{1/2}\right) \xrightarrow{\ell\to\infty} 0$$

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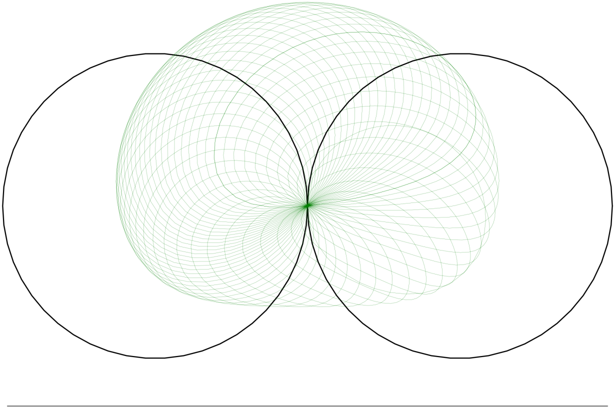

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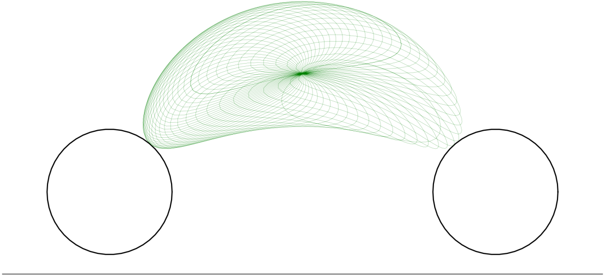

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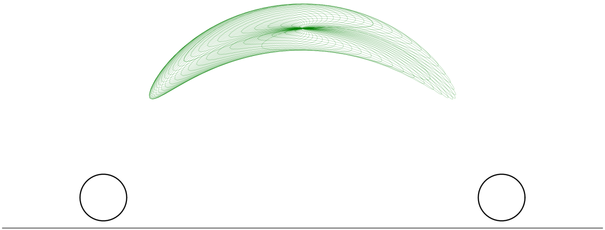

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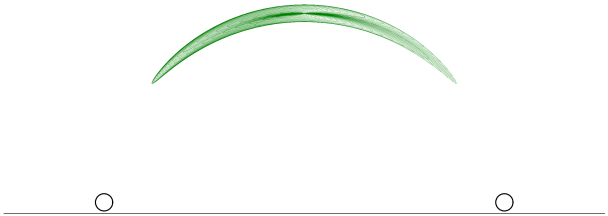

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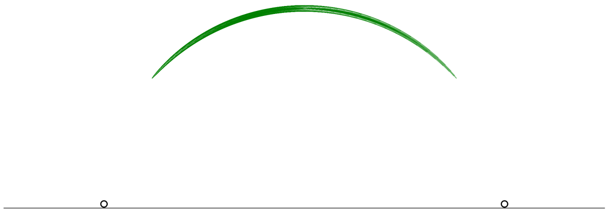

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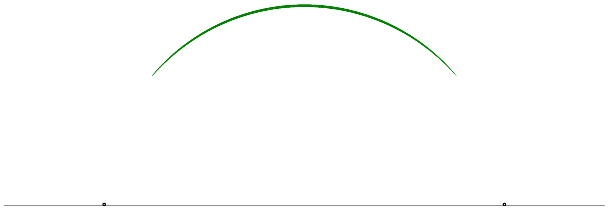

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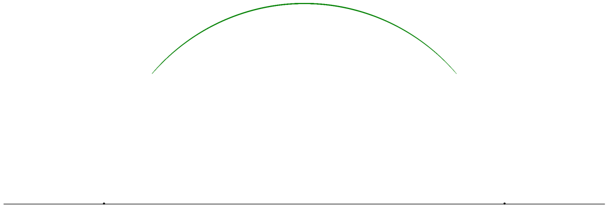

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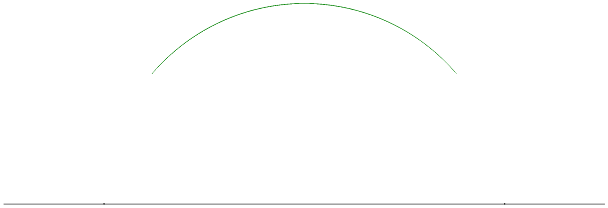

geodesics

horocycles

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- Through every tangent vector there are two horocycle arcs.

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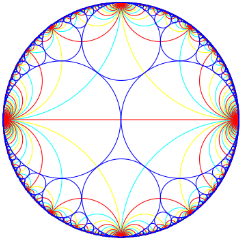

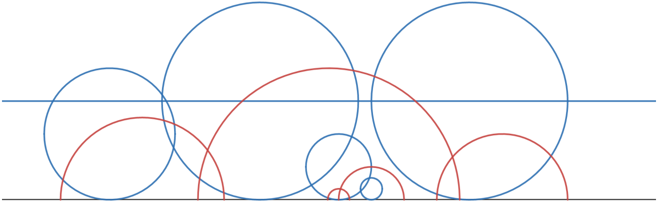

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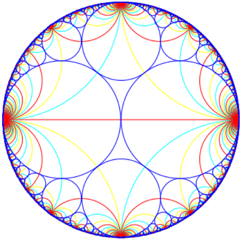

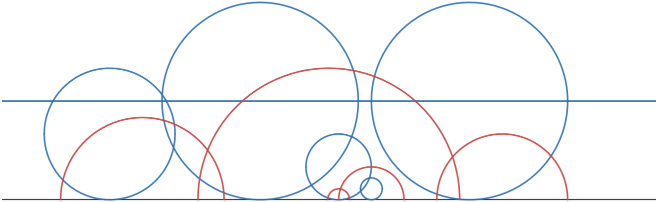

Horocyclic Brunn-Minkowski

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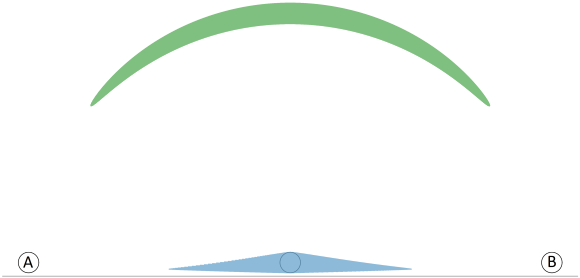

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Theorem (A., Klartag '22)

$A, B \subseteq \mathbf{H}$ *Borel, nonempty,* $0 < \lambda < 1$,

$$\mathrm{Area}([A : B]^h_\lambda)^{1/2} \geq (1 - \lambda) \cdot \mathrm{Area}(A)^{1/2} + \lambda \cdot \mathrm{Area}(B)^{1/2}.$$

When A, B are concentric discs, or if A or B is a singleton, equality holds.

Horocyclic Borell-Brascamp-Lieb inquality

Theorem (A., Klartag '22)

Let $f, g, h : \mathbf{H} \to [0, \infty)$ *be measurable, with* f *and* g *integrable with a non-zero integral. Let* $0 < \lambda < 1$ *and* $p \in [-1/2, +\infty]$. *Assume that for any* $x, y \in \mathbf{H}$ *with* $f(x)g(y) > 0$,

$$h\left([x : y]_\lambda^h\right) \geq M_p(f(x), g(y)\,; \lambda), \qquad \textit{where}$$

$$M_p(a, b; \lambda) = \begin{cases} ((1-\lambda)a^p + \lambda b^p)^{1/p} & p \notin \{0, \pm\infty\} \\ a^{1-\lambda} b^\lambda & p = 0 \\ \max\{a, b\} & p = +\infty \\ \min\{a, b\} & p = -\infty. \end{cases}$$

Then

$$\int_{\mathbf{H}} h \geq M_{p/(1+2p)}\left(\int_{\mathbf{H}} f, \int_{\mathbf{H}} g\,; \lambda\right).$$

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- If $\gamma \in \Gamma$ then $\gamma^{t_0,\lambda} \in \Gamma$ for every $t_0 \in \mathbb{R}$ and $\lambda > 0$, where $\gamma^{t_0,\lambda}(t) = \gamma\left(\lambda(t - t_0)\right)$ and $I_{\gamma^{t_0,\lambda}} = \lambda^{-1} I_\gamma + t_0$.

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- For every $p, q \in M$ there exists at least one path $\gamma \in \Gamma$ such that $\gamma(t) = p$ and $\gamma(t') = q$ for some $t < t'$.
- Γ is projectively Finsler - metrizable.

Minkowski averaging with respect to a path space

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Γ *Path space on* M,

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Problem

Suppose that M is endowed with a measure μ with a smooth density. Under what conditions on Γ, μ and N does the above operation satisfy the Brunn-Minkowski inequality

$$\mu([A : B]_\lambda^\Gamma)^{1/N} \geq (1 - \lambda) \cdot \mu(A)^{1/N} + \lambda \cdot \mu(B)^{1/N}$$

for every A, B Borel, nonempty and every $0 < \lambda < 1$?

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  Then there exists a function $\kappa : M \to \mathbb{R}$ *such that* Γ *is the set of solutions to the equation* $\nabla_{\dot\gamma}\dot\gamma = \kappa(\dot\gamma)|\dot\gamma|\dot\gamma^\perp$*, and*

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- *Suppose that Γ has the form above. Then $(\star)$ holds locally: for every $p \in M$ there exists a neighborhood $U \ni p$ such that $(\star)$ holds for every $A, B \subseteq U$ Borel, nonempty.*

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Strategy: localization ("needle decomposition") - Klartag '14, Payne - Weinberger '60 , Gromov - Milman '87, Lovasz - Simonovitz '93.

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3. ν is a measure on Λ.

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&= \left((1-\lambda)\cdot\mathrm{Area}(A)^{1/2} + \lambda\cdot\mathrm{Area}(B)^{1/2}\right)^2.
\end{aligned}$$

Needle decomposition

So we need:

1. A disintegration of measure:

$$\mathrm{Area}(S) = \int_\Lambda \mu_\gamma(S) d\nu(S) \qquad \text{for all } S \text{ Borel}$$

2. Mass balance:

$$\frac{\mu_\gamma(A)}{\mu_\gamma(B)} = \frac{\mathrm{Area}(A)}{\mathrm{Area}(B)} \qquad \text{for } \nu \text{ - a.e. } \gamma \in \Lambda.$$

3. Needlewise Brunn-Minkowski:

$$\mu_\gamma([A:B]^h_\lambda)^{1/2} \geq (1-\lambda) \cdot \mu_\gamma(A)^{1/2} + \lambda \cdot \mu_\gamma(B)^{1/2} \qquad \gamma \in \Lambda.$$

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In general, not every path space can be projectively Finsler - metrized. In dimension 2 this is possible locally.

Needlewise Brunn-Minkowski

Lemma

Let $F : [0,T] \times (-\varepsilon, \varepsilon) \to \mathbf{H}$ *be a locally Lipschitz map such that* $\det dF \neq 0$ *a.e., and for a.e. every* $s \in (-\varepsilon, \varepsilon)$*, the curve* $t \mapsto F(t,s)$ *is a constant-speed oriented horocycle.*

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is affine-linear for almost every $s \in (-\varepsilon, \varepsilon)$*. Here* $\det$ *is with respect to the Euclidean area form on* $[0,T] \times (-\varepsilon, \varepsilon)$ *and the hyperbolic area form on* $\mathbf{H}$*.*

Corollary

Each needle μ_γ *is given by* $\mu_\gamma = \gamma_\#(m_\gamma)$ *for some measure* m_γ *with an affine density on an interval* $I \subseteq \mathbb{R}$*.*

Needlewise Brunn-Minkowski

Theorem (Borell '75)

Let m be a Borel measure on an interval I. Suppose that m has a concave density with respect to the Lebesgue measure. Then m is 1/2-concave, i.e.

$$m((1-\lambda)A+\lambda B)^{1/2} \geq (1-\lambda)\cdot m(A)^{1/2}+\lambda\cdot m(B)^{1/2}$$

for every $A, B \subseteq I$ Borel, nonempty.

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Corollary (up to orientation issues)

For ν-a.e. $\gamma \in \Lambda$,

$$\mu_\gamma([A:B]^h_\lambda)^{1/2} \geq (1-\lambda)\cdot \mu_\gamma(A)^{1/2} + \lambda\mu_\gamma(B)^{1/2}.$$

Thank you!