On the dimensional Brunn-Minkowski conjecture: the role of symmetry

(based on the joint work with Alexander Kolesnikov.)

Galyna V. Livshyts

Georgia Institute of Technology

UGA
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Brunn-Minkowski inequality

Recall: Minkowski's sum of arbitrary sets K and L in $\mathbb{R}^n$

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Equivalently, the (apriori stronger) additive form:

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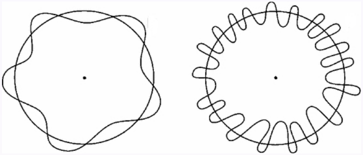
- Brunn-Minkowski inequality constitutes a fundamental concavity property of Lebesgue measure in $\mathbb{R}^n$.
- Impies Young's convolution inequality;
- Is a fundamental tool in convexity (duality&volumes, sections of convex bodies, projections of convex bodies, upper estimates on difference bodies, center of mass, coverings);
- Is a fundamental tool for obtaining concentration properties in probability;
- Is a fundamental tool in PDE thanks to its equality cases characterizations...

Relations of Brunn-Minkowski inequality to the isoperimetric inequality

Isoperimetric inequality

For any K such that $|K| = |B_2^n|$ we have $|\partial K|_{n-1} \geq |\partial B_2^n|_{n-1}$.



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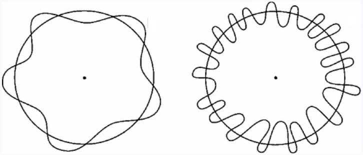

Brunn-Minkowski $\rightarrow$ Isoperimetric inequality

$$|\partial K|_{n-1} = \lim_{\epsilon \to 0} \frac{|K + \epsilon B_2^n| - |K|}{\epsilon} \geq \lim_{\epsilon \to 0} \frac{\left(|K|^{\frac{1}{n}} + \epsilon |B_2^n|^{\frac{1}{n}}\right)^n - |K|}{\epsilon}$$

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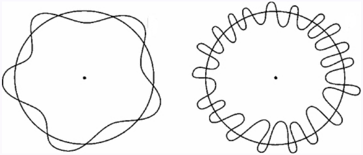

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and hence

$$\frac{|\partial K|_{n-1}}{|K|^{\frac{n-1}{n}}} \geq \frac{|\partial B_2^n|_{n-1}}{|B_2^n|^{\frac{n-1}{n}}}.$$

More generally: log-concavity

Log-concave functions

A function is called log-concave if its logarithm is concave, i.e. $f(\lambda x+(1-\lambda)y) \geq f(x)^{\lambda} f(y)^{1-\lambda}$.

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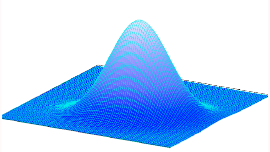
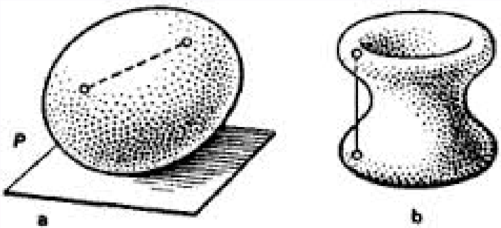
- Gaussian measure γ with density $\frac{1}{\sqrt{2\pi}^n}e^{-\frac{|x|^2}{2}}$;
- Lebesgue measure;
- Poisson density...



Preliminaries

- A convex body in $\mathbb{R}^n$ is a convex set with non-empty interior.



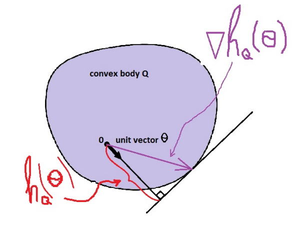
- They shall be usually denoted K, L.
- We shall usually assume that they contain the origin.
- A body K is called symmetric if $x \in K \implies -x \in K$.

Preliminaries

- Support function $h_K : \mathbb{R}^n \to \mathbb{R}^+$ of a convex body K is defined

$$h_K(x) = \max_{y \in K} \langle x, y \rangle;$$

If $u \in \mathbb{S}^{n-1}$ then $h_K(u)$ is the distance from the origin to the support hyperplane to K, orthogonal to u.



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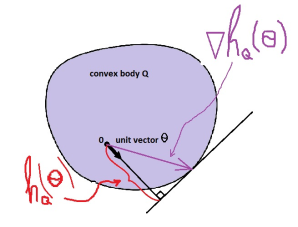

- $h_{K+L} = h_K + h_L$, $h_{\lambda K} = \lambda h_K$.

Brunn-Minkowski inequality is equivalent to its local form

Claim

Fix a convex body K with support function h, and pick an arbitrary function $\psi : \mathbb{S}^{n-1} \to \mathbb{R}$. Consider a family of convex bodies K_s with support functions $h_s = h + s\psi$. Set $F(s) = |K_s|$. Then

$$|\lambda K + (1-\lambda)L|^{\frac{1}{n}} \geq \lambda |K|^{\frac{1}{n}} + (1-\lambda)|L|^{\frac{1}{n}}$$

is equivalent to

$$F(0)F''(0) - \frac{n-1}{n}F'(0)^2 \leq 0.$$

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Analogously, log-concavity of F at $s=0$ is equivalent to the multiplicative form of Brunn-Minkowski inequality.

Brunn-Minkowski inequality in $\mathbb{R}^2$ for convex sets: relations to Poincare inequality

In the case $n = 2$,

$$|K| = \frac{1}{2}\int_{-\pi}^{\pi} h_K(h_K + \ddot{h_K}) = \frac{1}{2}\int_{-\pi}^{\pi} h_K^2 - \dot{h_K}^2$$

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and $\frac{1}{2}$-concavity of F

$$F(0)F''(0) - \frac{1}{2}F'(0)^2 \leq 0$$

writes as

$$\left(\int h^2 - \dot{h}^2\right) \cdot \left(\int \psi^2 - \dot{\psi}^2\right) - \left(\int h\psi - \dot{h}\dot{\psi}\right)^2 \leq 0.$$

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Note: if $h=1$ (corresponds to perturbing the unit ball), (10) becomes the Poincare inequality:

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$$\left(\hat{h}(0)^2 - \sum_{k\neq 0}(k^2-1)\hat{h}(k)^2\right)\left(\hat{\psi}(0)^2 - \sum_{k\neq 0}(k^2-1)\hat{\psi}(k)^2\right) \tag{11}$$

$$\leq \left(\hat{h}(0)\hat{\psi}(0) - \sum_{k\neq 0}(k^2-1)\hat{h}(k)\hat{\psi}(k)\right)^2.$$

(11) can be verified directly! BUT: killing $k=1$ does not help:(

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Question

How does Brunn-Minkowski inequality improve under the symmetry and convexity assumptions?

Log-Brunn-Minkowski conjecture

Geometric average of convex bodies

$$\lambda K +_0 (1-\lambda)L := \{x \in \mathbb{R}^n : \langle x, u\rangle \leq h_K^{\lambda}(u) h_L^{1-\lambda}(u) \; \forall u \in \mathbb{S}^{n-1}\}.$$

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Log-Brunn-Minkowski Conjecture (Böröczky, Lutwak, Yang, Zhang, 2011)

Let $n \geq 2$ be an integer. Let K and L be symmetric convex sets in $\mathbb{R}^n$. Then

$$|\lambda K +_0 (1-\lambda)L| \geq |K|^{\lambda}|L|^{1-\lambda}.$$

Stronger than the Brunn-Minkowski inequality by arithmetic-geometric mean inequality.

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Stronger than the Brunn-Minkowski inequality by arithmetic-geometric mean inequality.

- True for $n = 2$ (Stancu; Böröczky, Lutwak, Yang and Zhang)
- True for unconditional sets (i.e. symmetric with respect to every coordinate hyperplane) (Saroglou; Cordero-Erasquin, Fradelizi, Maurey)
- True for complex convex bodies (Rotem)
- True in a neighborhood of a Euclidean ball (Colesanti, L, Marsiglietti; improved in Colesanti, L)
- Works well with the L_2-method (Kolesnikov-Milman)

Böröczky, Colesanti, Cordero, Fradelizi, Henk, Huang, Hug, Linke, Lutwak, Marsiglietti, Morey, Oliker, Saraglou, Stancu, Vikram, Xu, Yang, Zhang...

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Gardner-Zvavitch conjecture, 2007

Let γ be the Gaussian measure (more generally, even log-concave measure), and K and L be symmetric convex bodies. Then

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- When sets contain origin: not necessarily true (Tkocz, Nayar);

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- Does not imply/ does not follow from Ehrhard's inequality;
- Follows from Log-Brunn-Minkowski conjecture! Hence true in dimension 2 and for unconditional sets. (L, Marsiglietti, Nayar, Zvavitch).
- Is a bit nicer than Log BM since we are dealing with Minkowski sum.

Theorem about the Gaussian measure

Suppose γ is the Gaussian measure on $\mathbb{R}^n$.

Theorem (Kolesnikov, L 2018+)

For **gaussian barycentered** convex sets K and L, and for any $\lambda \in [0,1]$, we have

$$\gamma(\lambda K+(1-\lambda)L)^{\frac{1}{2n}} \geq \lambda\gamma(K)^{\frac{1}{2n}}+(1-\lambda)\gamma(L)^{\frac{1}{2n}}.$$

Theorem general

Theorem (Kolesnikov, L 2018+)

Let γ be a log-concave measure on $\mathbb{R}^n$ with density $e^{-V(x)}$, for some **even** convex function $V : \mathbb{R}^n \to \mathbb{R}$. We shall assume that $k_1, k_2 > 0$ are such constants that

$$\nabla^2 V \geq k_1 Id; \; \Delta V \leq k_2 n.$$

Let $R = \frac{k_2}{k_1} \geq 1$. For any pair of **symmetric** convex sets K and L, and for any $\lambda \in [0,1]$, one has

$$\gamma(\lambda K + (1-\lambda)L)^{\frac{C}{n}} \geq \lambda \gamma(K)^{\frac{C}{n}} + (1-\lambda)\gamma(L)^{\frac{C}{n}}, \tag{12}$$

where

$$C = C(R) = \frac{2}{(\sqrt{R}+1)^2}.$$

Replace symmetry with something weaker

In fact, we get a bound under a weaker than symmetry assumption:

Theorem (Kolesnikov, L 2018+)

Suppose μ is log-concave. For any pair of convex sets K and L which satisfy

$$\int_K \nabla V d\mu = \int_L \nabla V d\mu = 0,$$

and for any $\lambda \in [0,1]$, one has

$$\mu(\lambda K + (1-\lambda)L)^{\frac{c'}{n}} \geq \lambda \mu(K)^{\frac{c'}{n}} + (1-\lambda)\mu(L)^{\frac{c'}{n}}, \tag{13}$$

where

$$c' = c'(R) = \frac{1}{R+1} > 0.$$

Definitions (GAUSSIAN CASE)

Gardner-Zvavitch constant

We shall define the Gardner-Zvavitch constant C_0 to be the largest number so that for all *barycentered* convex sets K, L, and for any $\lambda \in [0,1]$

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Weighted Laplace operator

$$Lu = \Delta u - \langle \nabla u, x \rangle. \tag{15}$$

Integration by parts:

$$\int_{\mathbb{R}^n} v \cdot Lu d\gamma = -\int_{\mathbb{R}^n} \langle x, \nabla u \rangle d\gamma.$$

Steps of the proof (GAUSSIAN CASE)

Step 1

Let C_1 to be the largest number, such that for every $u \in C^2(K)$ with $Lu = 1$ on K,

$$\frac{1}{\gamma(K)} \int_K ||\nabla^2 u||^2 + |\nabla u|^2 d\gamma \geq \frac{C_1}{n}.$$

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Claim

As before, let $F(s)=\gamma(K_s)$, where K_s has support function $h+s\psi$;

$$\gamma(\lambda K+(1-\lambda)L)^{\frac{1}{n}} \geq \lambda\gamma(K)^{\frac{1}{n}}+(1-\lambda)\gamma(L)^{\frac{1}{n}}$$

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$$F''(0)=\int_{\partial K}\left(H_x f^2-\langle \mathrm{II}^{-1}\nabla_{\partial K}f,\nabla_{\partial K}f\rangle\right)d\gamma_{\partial K}(x).$$

Here II is the second quadratic form of ∂K and

$$H_x=tr\mathrm{II}-\langle x,n_x\rangle.$$

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Second derivative

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$$\int_K (Lu)^2 d\gamma(x) = \int_K ||\nabla^2 u||_{HS}^2 + |\nabla u|^2 d\gamma(x) + \tag{17}$$

$$\int_{\partial K} H_x f^2 - 2\langle \nabla_{\partial K} u, \nabla_{\partial K} f \rangle + \langle \mathrm{II} \nabla_{\partial K} u, \nabla_{\partial K} u \rangle d\gamma_{\partial K}(x).$$

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For a positive-definite matrix A,

$$\langle Ax, x \rangle + \langle A^{-1} y, y \rangle \geq 2\langle x, y \rangle. \tag{18}$$

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Neumann system

We can solve the Neumann system and find such $u : K \to \mathbb{R}$ that

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and have additionally that

$$Lu = 1, \tag{20}$$

provided that

$$\int_{\partial K} f d\gamma_{\partial K} = \gamma(K).$$

Combining all of the above, we note that the conjecture of Gardner and Zvavitch follows from

$$\frac{1}{\gamma(K)} \int_K ||\nabla^2 u||^2_{HS} + |\nabla u|^2 d\gamma(x) \geq \frac{C_0}{n}. \tag{21}$$

That finishes the proof of Step 1.

Step 2

Recall the statement of Step 2:

For all u with $Lu = 1$ on K,

$$\int_K ||\nabla^2 u||_{HS}^2 + |\nabla u|^2 d\gamma(x) \geq \int_K \frac{1}{|x|^2 + n} d\gamma.$$

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- Using Cauchy inequality we bound it from below by

$$\frac{1}{n} \int \frac{n}{n + |x|^2} \square.$$

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Lemma

For any barycentered convex body K,

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$$\frac{1}{\gamma(K)}\int_K \frac{1}{\frac{|x|^2}{n}+1}d\gamma \geq \frac{1}{2}.$$

Proof: By Jensen's inequality,

$$\frac{1}{\gamma(K)}\int_K \frac{1}{\frac{|x|^2}{n}+1} \geq \frac{1}{\frac{1}{\gamma(K)}\int_K \frac{|x|^2}{n}dx+1} \geq \frac{1}{2}.\square$$

Towards sharper bounds?

Question

Given symmetric convex K, does there exist a function $F : K \to \mathbb{R}$ such that for all $u : K \to \mathbb{R}$ with $Lu = F$ we have

$$\int_K (||\nabla^2 u||_{HS}^2 + |\nabla u|^2) d\gamma(x) \geq \tag{24}$$

$$\int_K F^2 d\gamma(x) - \frac{n - c}{n\gamma(K)} \left(\int_K F d\gamma(x) \right)^2 ?$$

Ideally with $c = 1$?

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Specific choice

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Proof: Note that (25) rewrites:

$$n\gamma(K) + \int_K x^2 d\gamma \geq n^2\gamma(K) - 2n \int_K x^2 d\gamma + \int_K x^4 d\gamma \tag{26}$$

$$- \left(n^2\gamma(K) - 2n \int_K x^2 d\gamma + \frac{1}{\gamma(K)} \Big(\int_K x^2 d\gamma \Big)^2 \right)$$

$$+ \frac{1}{n} \left(n^2\gamma(K) - 2n \int_K x^2 d\gamma + \frac{1}{\gamma(K)} \Big(\int_K x^2 d\gamma \Big)^2 \right).$$

Case of dilates

- Rearranging, we get

$$\left[\int_K x^4 d\gamma - \frac{1}{\gamma(K)}\Big(\int_K x^2 d\gamma\Big)^2 - 2\int_K x^2 d\gamma\right] + \tag{27}$$
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- Recall the B-Theorem of Cordero-Erasquin, Fradelizi, Maurey:

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Corollary

When $K = tL$, the conjecture of Garnder and Zvavitch follows.

A stronger statement in the Gaussian case!

More news in the Gaussian case

For convex sets K and L containing the origin, and for any $\lambda \in [0,1]$, we have

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Indeed, the function $\gamma(tK)$ is increasing, and $\gamma(tK)'_{t=0} \geq 0$ implies (30).

Thanks for your attention!