Minkowski's Theorem for positively concave and positively homogenous measures and it's applications to measure comparison of convex bodies.

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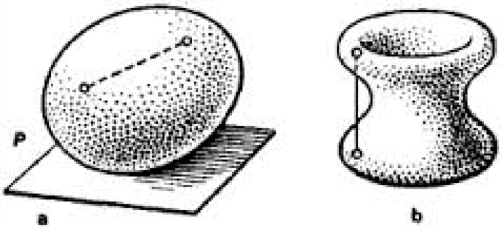
Preliminaries

- A convex body in $\mathbb{R}^n$ is a convex set with non-empty interior.



- They shall be usually denoted K, L.
- We shall usually assume that they contain the origin.
- A body K is called symmetric if $x \in K \implies -x \in K$.
- A convex body K is called strictly convex if its boundary contains no interval.

Preliminaries

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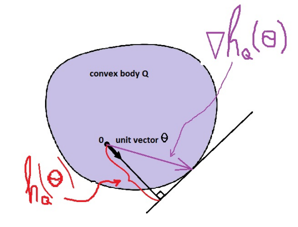
- Support function $h_K : \mathbb{R}^n \to \mathbb{R}^+$ of a convex body K is defined

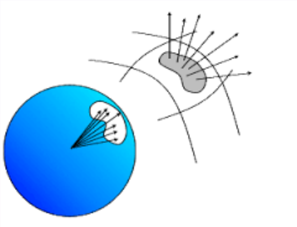
$$h_K(x) = \max_{y \in K} \langle x, y \rangle;$$

  If $u \in \mathbb{S}^{n-1}$ then $h_K(u)$ is the distance from the origin to the support hyperplane to K, orthogonal to u.
- $\nabla h_K(u)$ is the vector at which the support hyperplane touches ∂K.



Preliminaries

- The Gauss map $\nu_K : \partial K \to \mathbb{S}^{n-1}$ corresponds $x \in \partial K$ to the set of its normals n_x.



- If the set K is C^2-smooth (i.e., its support function is C^2) and strictly convex then its Gauss map is $1-1$.

Preliminaries

- The surface area measure σ_K of a convex body K is the push-forward of the Hausdorff $(n-1)$-dimensional measure on ∂K to the sphere:

$$\sigma_K(\Omega) = \int_{\nu_K^{-1}(\Omega)} dH_{n-1}(x).$$

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- If K is strictly convex and C^2-smooth then c_K has density $\frac{1}{n}h_K f_K$.

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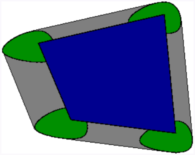
- The mixed volume of convex bodies K and L:

$$V_1(K,L) = \frac{1}{n} \liminf_{\epsilon \to 0} \frac{|K+\epsilon L| - |K|}{\epsilon}$$



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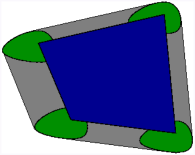

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- Brunn-Minkowski inequality

$$|\lambda K + (1-\lambda)L|^{\frac{1}{n}} \geq \lambda |K|^{\frac{1}{n}} + (1-\lambda)|L|^{\frac{1}{n}},$$

- Which implies Minkowski's first inequality

$$V_1(K,L) \geq |K|^{\frac{n-1}{n}} |L|^{\frac{1}{n}}.$$

The definition of the weighted surface area measure

Definition

Let K be a convex set and μ be a measure on $\mathbb{R}^n$ with density $g(x)$. The surface area measure $\sigma_{\mu,K}$ of a convex body K with respect to μ is defined:

$$\sigma_{\mu,K}(\Omega) = \int_{\nu_K^{-1}(\Omega)} g(x) dH_{n-1}(x).$$

Minkowski's theorem

Minkowski's existence theorem

Let φ be a measure on $\mathbb{S}^{n-1}$, not supported on any great subsphere and barycentred at the origin. Then there exists a convex body K so that $\varphi = \sigma_K$; moreover, a convex body is determined uniquely (up to a shift) by its surface area measure.

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Is the same true about cone volume measure? I.e., does $h_K(u)d\sigma_K(u) = h_L(u)d\sigma_L(u)$ imply $K = L$?

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- Existence: in the symmetric case – yes (Böröczky, Lutwak, Yang, Zhang);
- Uniqueness: not true in $\mathbb{R}^2$ **only** for parallelograms with parallel sides (Stancu; Cage; Böröczky, Lutwak, Yang, Zhang);

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- A lot is still unknown.

Andrews, Böröczky, Cage, Chou, Colesanti, Cordero, Fradelizi, Gardner, Henk, Huang, Hug, Linke, Liu, Lutwak, Ludwig, Marsiglietti, Morey, Nayar, Oliker, Saraglou, Stancu, Tkozsh, Vikram, Xu, Wang, Yang, Zhang, Zhu, Zvavitch...

Measures with positive concavity and homogeneity

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- Emanuel Milman and Liran Rotem studied such measures and their isoperimetric properties.

Minkowski's theorem for measures with positive concavity and homogeneity

Theorem (L. 2016)

Let μ on $\mathbb{R}^n$ be a measure and $g(x)$ be its even $r-$*homogenous* density for some $r > -n$, and the restriction of g to some half space is $p-$*concave* for a $p > 0$. Let $\varphi(u)$ be an arbitrary even measure on $\mathbb{S}^{n-1}$, not supported on any great subsphere, such that $supp(\varphi) \subset int(supp(g)) \cap \mathbb{S}^{n-1}$. Then there exists a symmetric convex body K in $\mathbb{R}^n$ such that

$$d\sigma_{K,\mu}(u) = d\varphi(u).$$

Moreover, such convex body is determined uniquely up to a set of $\mu-$measure zero.

A weaker statement then the Log-Minkowski problem

Proposition

Let K and L be two symmetric, C^2*smooth*, strictly-convex bodies in $\mathbb{R}^n$ with support functions h_K and h_L and curvature functions f_K and f_L such that

$$\frac{\partial h_K(u)}{\partial e_1} f_K(u) = \frac{\partial h_L(u)}{\partial e_1} f_L(u)$$

for every $u \in \mathbb{S}^{n-1}$. Then $K = L$.

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Could one get $h_K(u) f_K(u) = h_L(u) f_L(u)$ for every $u \in \mathbb{S}^{n-1} \implies$

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for every $u \in \mathbb{S}^{n-1}$?

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Let K and L be symmetric convex bodies in $\mathbb{R}^n$. Suppose in every direction θ, $|K|\theta^{\perp}|_{n-1} \leq |L|\theta^{\perp}|_{n-1}$. Does it imply that $|K|_n \leq |L|_n$?

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A convex body L is called a projection body if for some other convex body Q,

$$h_L(u) = |Q|u^{\perp}|.$$

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Ball, Dann, Gardner, Giannopolus, Goodey, Hug, Koldobsky, Ludwig, Petti, Ryabogin, Schuster, Schneider, Schlumprecht, Zvavitch, Yaskin, Yaskina, Zhang,....

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$$p_{\mu,K}(\theta,t) := \frac{n}{2}\int_{\mathbb{S}^{n-1}} |\langle \theta,u\rangle| d\sigma_{\mu,tK}(u). \tag{1}$$

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$$P_{\lambda,K}(\theta) = |K|\theta^{\perp}|_{n-1}.$$

Shephard's problem for positively concave and positively homogenous measures

Theorem (L. 2016)

Fix $n \geq 1$, and consider $g : \mathbb{R}^n \to \mathbb{R}^+$, a function with a positive degree of concavity and a positive degree of homogeneity. Let μ be the measure on $\mathbb{R}^n$ with density $g(x)$.

1. Let K and L be symmetric strictly convex bodies, and let L additionally be a projection body. Assume that for every $\theta \in \mathbb{S}^{n-1}$ we have
$$P_{\mu,K}(\theta) \leq P_{\mu,L}(\theta).$$
Then $\mu(K) \leq \mu(L)$.
2. If in addition we assume that the support of g is a half-space, then for each symmetric convex body L which is not a projection body, there exists a symmetric convex body K such that for every $\theta \in \mathbb{S}^{n-1}$ we have
$$P_{\mu,K}(\theta) \leq P_{\mu,L}(\theta),$$
but $\mu(K) > \mu(L)$.

Tools

Mixed measure

Given sets K and L, and a measure *mu* on $\mathbb{R}^n$, we define their **mixed** $\mu-$**measure** as follows.

$$\mu_1(K,L) := \liminf_{\epsilon\to 0} \frac{\mu(K+\epsilon L)-\mu(K)}{\epsilon}.$$

We also introduce the following analogue of mixed volume:

$$V_{\mu,1}(K,L) = \int_0^1 \mu_1(tK,L)dt.$$

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Lemma (E. Milman, L. Rotem)

If μ has a p-concave $\frac{1}{p}$-homogenous density, then for $q = \frac{1}{n+\frac{1}{p}}$,

$$\mu_1(K, L) \geq \frac{1}{q}\mu(K)^{1-q}\mu(L)^q.$$

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Dual isoperimetric inequality

Let a measure μ be log-concave. Then for every pair of Borel sets K and L such that $\mu(K) = \mu(L)$, one has

$$\mu_1(K, L) \geq \mu_1(K, K).$$

Shephard for measures: part of the proof.

Proposition

L – projection body; for every $\theta \in \mathbb{S}^{n-1}$, $P_{K,\mu}(\theta) \leq P_{L,\mu}(\theta)$. Then $\mu(K) \leq \mu(L)$.

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L – projection body; for every $\theta \in \mathbb{S}^{n-1}$, $P_{K,\mu}(\theta) \leq P_{L,\mu}(\theta)$. Then $\mu(K) \leq \mu(L)$.

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Let K and L be two symmetric, C^2, strictly-convex bodies in $\mathbb{R}^n$ with support functions h_K and h_L and curvature functions f_K and f_L such that

$$\frac{\partial h_K(u)}{\partial e_1} f_K(u) = \frac{\partial h_L(u)}{\partial e_1} f_L(u)$$

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Proof of the (very) weak Log-Minkowski

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Thanks for your attention!