Maximal surface area of a convex set in $\mathbb{R}^n$ with respect to log concave rotation invariant measures.

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Isoperimetric Inequality

Let Q be a set in $\mathbb{R}^n$. Denote the boundary of Q by ∂Q. Let $|Q| = 1$. Question: How to minimize $|\partial Q|$ (the surface area of Q)?

Classical Isoperimetric Inequality

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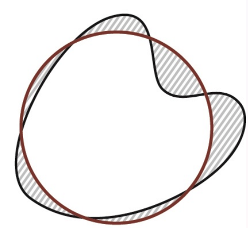

But if we replace the usual Lebesgue volume measure with another measure, the answer to that question may change!

Gaussian Measure

We recall, that the Standard Gaussian Measure γ_2 on $\mathbb{R}^n$ is the probability measure with density

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There is a convenient integral expression for $\gamma_2(\partial Q)$:

$$\gamma_2(\partial Q) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{\partial Q} e^{-\frac{|y|^2}{2}} d\sigma(y),$$

where $d\sigma(y)$ stands for Lebesgue surface measure.

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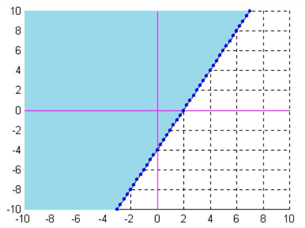

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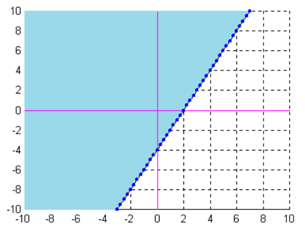

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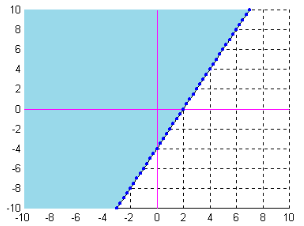

How to ask the reverse question?

By $\mathcal{K}_n$ we denote the set of all convex bodies in $\mathbb{R}^n$.
Let Q run over $\mathcal{K}_n$. What is the maximal Gaussian surface area of Q?

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Are there any other interesting measures for which it is natural to ask Isoperimetric type inequalities?

Definition of log concave measures

A Borel measure μ on $\mathbb{R}^n$ is called log concave, if for any compact sets $A, B \subset \mathbb{R}^n$ and for any $\lambda \in [0,1]$,

$$\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^{\lambda} \cdot \mu(B)^{1-\lambda}.$$

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Borell's Theorem

A measure is log-concave if and only if it has a density, and this density is a log concave function.

Kannan-Lovaszh-Simonovizh conjecture

The KLS conjecture suggests that for any convex body K,

$$\inf_{A\subset\mathbb{R}^n} \frac{|\partial A\cap K|}{|A\cap K|\cdot|K\setminus A|}$$

is attained on a certain halfspace up to a constant.

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We recall, that a measure is called isotropic if it is centred and the covariance matrix is unit. Any measure can be brought to an isotropic position via linear change of variables. For an isotropic probability log concave measure, the surface area of a proper halfspace is of a constant order.

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The restatement of Kannan-Lovaszh-Simonovizh conjecture

There exist an absolute constant $C>0$, such that for any isotropic probability log concave measure γ on $\mathbb{R}^n$ and for any $A\subset\mathbb{R}^n$ with $\gamma(A)\leq\frac{1}{2}$,

$$\frac{\gamma(\partial A)}{\gamma(A)}\geq C.$$

The best known result for the KLS conjecture is the following

Theorem (Ronen Eldan)

There exist an absolute constant $C > 0$, such that for any isotropic log concave measure γ on $\mathbb{R}^n$ and for any $A \subset \mathbb{R}^n$ with $\gamma(A) \leq \frac{1}{2}$,

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He proved as well, that the thin shell conjecture implies KLS conjecture up to a log factor.

Rotation invariant Log concave measures

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For rotation-invariant log concave measures KLS conjecture holds true. This is implied by the results of

- S. G. Bobkov, *Spectral gap and concentration for some spherically symmetric probability measures,* Lect. Notes Math. 1807 (2003), 37-43.
- M. Ledoux, *Spectral gap, logarithmic Sobolev constant, and geo- metric bounds.* Surveys in differential geometry. Vol. IX, 219-240, Surv. Differ. Geom., IX, Int. Press, Somerville, MA, 2004.

Question (generalization of Ball-Nazarov Theorems)

Fix a log concave rotation invariant measure γ on $\mathbb{R}^n$ with density $C_n e^{-\varphi(|y|)}$ on $\mathbb{R}^n$. Let Q be a convex body in $\mathbb{R}^n$. What are the bounds for $\max \gamma(\partial Q)$?

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For any positive p

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For $p \geq 1$ the measure γ_p is log concave, but for $p < 1$ it is not.

The reverse isoperimetric inequality for Rotation invariant Log concave measures. The main result.

Theorem (G. L. 2013)

Fix $n \geq 2$. Let γ be log concave rotation invariant measure on $\mathbb{R}^n$. Consider a random vector X in $\mathbb{R}^n$ distributed with respect to γ.

$$\max_{Q \in \mathcal{K}_n} \gamma(\partial Q) = C \frac{\sqrt{n}}{\sqrt{\mathbb{E}|X|} \sqrt[4]{Var|X|}},$$

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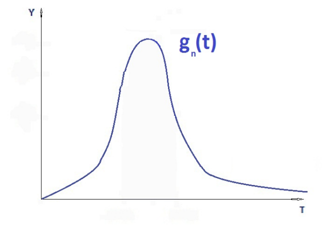

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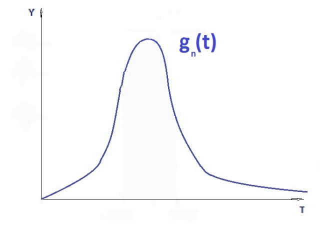

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The point of maxima

The mass of the integral

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For a random vector X distributed with respect to γ,

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Examples.

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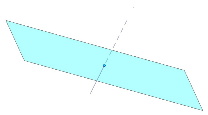

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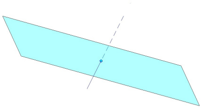

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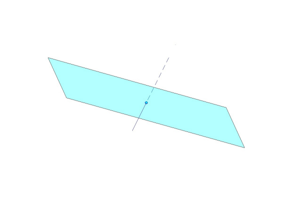

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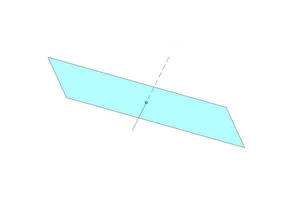

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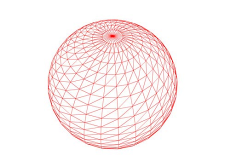
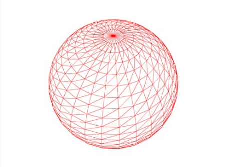


The measure of the maximal sphere



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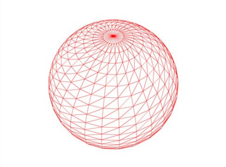
Let RS^{n-1} be a sphere of radius R centered at the origin.



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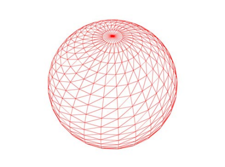

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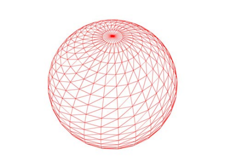

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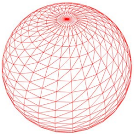

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$$\gamma(t_0 S^{n-1}) = \frac{g_{n-1}(t_0)}{J_{n-1}}.$$

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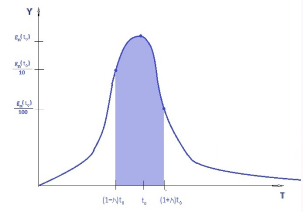

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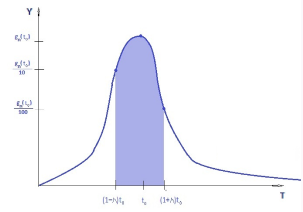

For a random vector X distributed with respect to γ,

$$Var|X| = C(\lambda t_0)^2.$$

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Fix $n \geq 2$. Let γ be a measure with density $C_n e^{-\varphi(|y|)}$. Let t_0 be the solution of $\varphi'(t)t = n-1$. Define $\lambda = \frac{\int_0^\infty t^{n-1} e^{-\varphi(t)} dt}{t_0^n e^{-\varphi(t_0)}}$. Then

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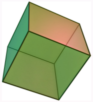

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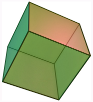

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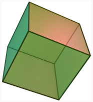

- A uniform measure on a thin spherical annulus

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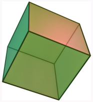

- A uniform measure on a thin spherical annulus is an example of rotation invariant but not log concave measure for which the above fails to be true.

Gaussian surface area of a polytope with K faces

Theorem (F. Nazarov)

Let $P = \cap_{i=1}^{K} \{\langle x, \theta_i \rangle \leq \rho_i\}$ be a polytope with at most K faces. Then

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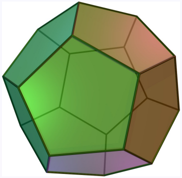
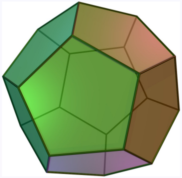
$$P = \cap_{i=1}^{K}\{\langle x, \theta_i \rangle \leq \rho\}.$$

On average, P is the desired polytope with the Gaussian surface area of order $\sqrt{\log K}$.



Polytopes



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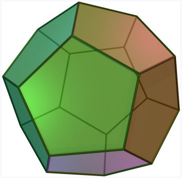
Fix $n \geq 2$. For a log concave rotation invariant measure γ on $\mathbb{R}^n$ and a random vector X distributed with respect to γ, fix an integer $K \leq e^{c\frac{\mathbb{E}|X|}{\sqrt{Var|X|}}}$.



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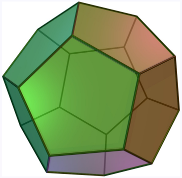

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where maximum runs over all polytopes P with at most K faces. More precisely,

$$c\frac{\sqrt{n}}{t_0}\min\left(\sqrt{\log K}, \frac{1}{\sqrt{\lambda}}\right) \leq \max_{Q\in\mathbb{P}_K} \gamma(\partial Q) \leq C\frac{\sqrt{n}}{t_0}\sqrt{\log\frac{K}{\log\frac{1}{\lambda\sqrt{\log K}}}\log\frac{1}{\lambda\sqrt{\log K}}}.$$

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Partial results: concentration

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Relation to surface area

For any convex set Q, for any $0 \leq h \leq \frac{4\sqrt{n}}{\sqrt{\pi}\gamma_2(\partial Q)}$ we have:

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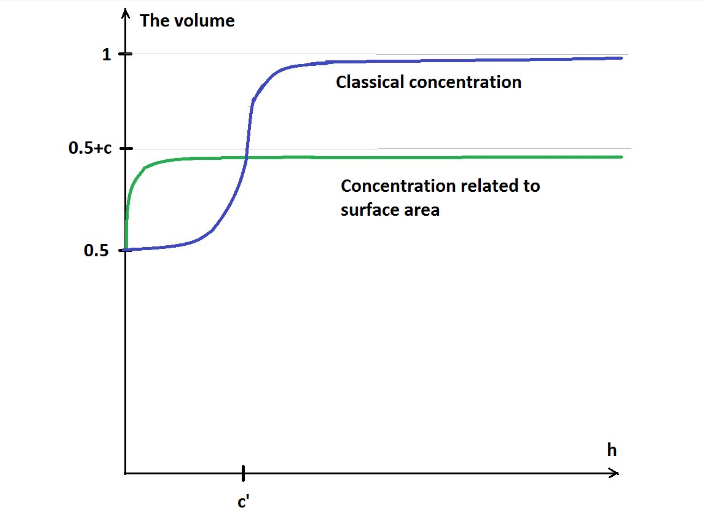
For all $h \leq 2\sqrt[4]{\pi}\frac{\gamma(\partial Q)}{n^{\frac{1}{4}}}$ the second estimate is better then the first one.

For example, for the sets of maximal surface area and of Gaussian measure $\frac{1}{2}$:



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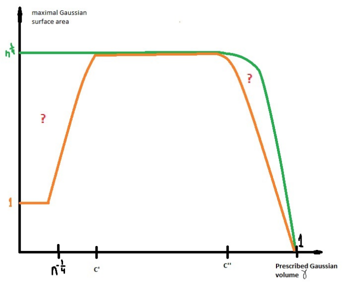

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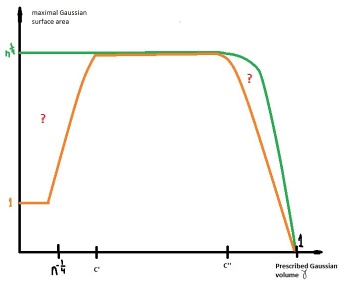

Thanks for your attention!