The smallest singular value of inhomogeneous random matrices and efficient net estimates

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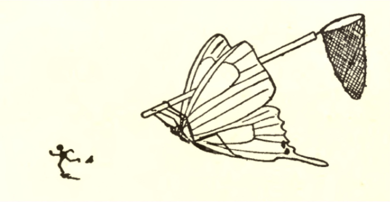

Notation

- $\mathbb{R}^n$ – euclidean n-dimensional space with standard basis $e_1, \dots, e_n$;
- B_2^n – euclidean unit ball in $\mathbb{R}^n$;
- $\mathbb{S}^{n-1}$ – unit sphere in $\mathbb{R}^n$;
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- $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$;
- The Hilbert-Schmidt norm of a matrix A is $||A||_{HS} = \sqrt{\sum_{i,j} a_{ij}^2}$;
- Singular values of A are the axi of the ellipsoid AB_2^n, denoted $\sigma_1(A) \geq \ldots \geq \sigma_n(A)$;
- The operator norm $||A|| = \sup_{x \in \mathbb{S}^{n-1}} |Ax| = \sigma_1(A)$;
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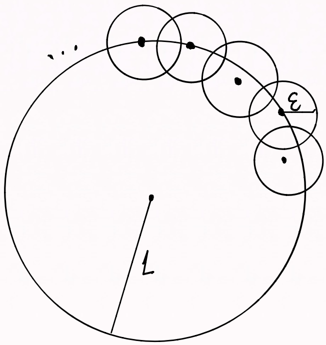
Notation / Preliminaries

Recall: there exists a Euclidean epsilon-net N on the unit sphere of cardinality < $(3/\varepsilon)^n$.



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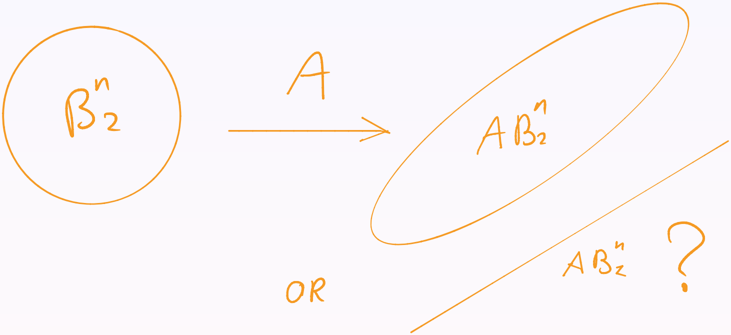

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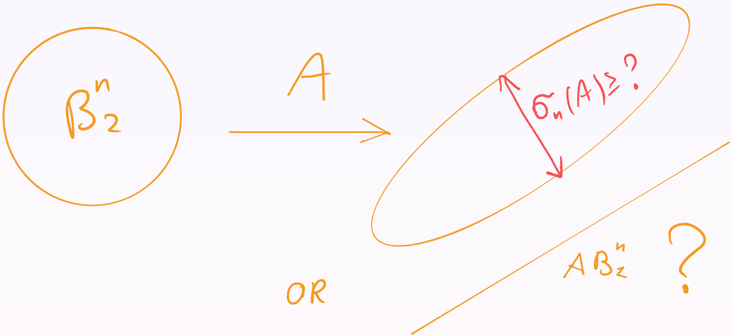

A harder question: how likely is the smallest singular value $\sigma_n(A) = \inf_{x \in \mathbb{S}^{n-1}} |Ax|$ to be bigger than  ?

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$$\sigma_n(A) \approx \frac{1}{\sqrt{n}}.$$

Furthermore, for every $\epsilon \in (0,1)$,

$$P\left(\sigma_n(A) \leq \frac{\epsilon}{\sqrt{n}}\right) \leq C\epsilon.$$

(Edelman, Szareck independently in 1990s)

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- Bourgain, Vu, Wood, 2010: $P(\sigma_n(A) = 0) \leq \sqrt{2}^{-n}$;
- Tikhomirov, 2019: $P(\sigma_n(A) = 0) \leq (0.5 + o(1))^n$!

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A is $n \times n$, has entries a_{ij} uniformly anti-concentrated, i.i.d., $\mathbb{E}a_{ij} = 0$, $\mathbb{E}a_{ij}^2 = 1$

Rebrova, Tikhomirov 2016:

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Remark

In fact, it is enough to assume for any $p > 0$,

$$\sum_{i=1}^{n}\left(\mathbb{E}|Ae_i|^{2p}\right)^{\frac{1}{p}} \leq Kn^2; \quad \sum_{i=1}^{n}\left(\mathbb{E}|A^T e_i|^{2p}\right)^{\frac{1}{p}} \leq Kn^2.$$

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Bai, Cook, Edelman, Gordon, Guedon, Huang, Koltchinckii, Latala, Litvak, Lytova, Meckes, Meckes, Mendelson, Pajor, Paouris, Rebrova, Rudelson, O'Rourke, Szarek, Tao, Tatarko, Tomczak-Jaegermann, Tikhomirov, Van Handel, Vershynin, Vu, Yaskov, Yin, Youssef,...

The smallest singular value: unstructured square case

Theorem (L, Tikhomirov, Vershynin 2019+)

Let A be an $n \times n$ random matrix with

- independent entries a_{ij}
- $\mathbb{E}||A||_{HS}^2 \le Kn^2$
- a_{ij} are UAC, that is $P(\sup_{z\in\mathbb{R}} |a_{ij} - z| < 1) < b \in (0,1)$

Then for every $\epsilon \in (0,1)$,

$$P\left(\sigma_n(A) < \frac{\epsilon}{\sqrt{n}}\right) \le C\epsilon + e^{-cn},$$

where C and c are absolute constants which depend (polynomially) only on K and b.

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Question: what if A is an $N \times n$ random matrix with $N \geq n$?

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$N \geq n$, a_{ij} i.i.d. sub-Gaussian, $\mathbb{E}a_{ij} = 0$, $\mathbb{E}a_{ij}^2 = 1$. Then for any $\epsilon \in (0,1)$,

$$P\left(\sigma_n(A) \leq \epsilon(\sqrt{N+1} - \sqrt{n})\right) \leq C_1 \epsilon^{N-n+1} + e^{-C_2 N};$$

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Replaced sub-Gaussian with $\mathbb{E}a_{ij}^{C_1} \leq 1$, but $N \in [n, n+C_2]$

Vershynin, 2011

Replaced sub-Gaussian with $\mathbb{E}a_{ij}^4 < \infty$ but

$$P\left(\sigma_n(A) \leq \epsilon(\sqrt{N+1} - \sqrt{n})\right) \leq \delta(\epsilon) \to_{\epsilon \to 0} 0.$$

Arbitrary aspect ratio

Theorem (L. 2018+)

Let $N \geq n \geq 1$ be integers. Let A be an $N \times n$ random matrix with

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Then for every $\epsilon > 0$,

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where C and c are absolute constants which depend (polynomially) only on the concentration function bounds.

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Remark: a more general result in fact follows...

Very tall case

Proposition 1 (L. 2018+) tall case with dependent columns

Suppose A is an $N \times n$ random matrix with independent rows, $\mathbb{E}||A||_{HS}^2 \leq KNn$, $N \geq C_0 n$, and assume for every $x \in \mathbb{S}^{n-1}$,

$$P(\sup_{y \in \mathbb{R}} |\langle A^T e_i, x\rangle - y| \leq 1) \leq b \in (0,1). \tag{1}$$

Then

$$\mathbb{E}\sigma_n(A) \geq c\sqrt{N}.$$

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Proposition 2 (L. 2018+) tall case with low moments

Fix $p > 0$. Suppose $N \geq C_0' n$, A is an $N \times n$ random matrix with independent UAC entries. Suppose

$$\sum_{i=1}^{n} \left(\mathbb{E}|Ae_i|^{2p}\right)^{\frac{1}{p}} \leq KnNe^{\frac{c_0 N}{n}}.$$

Then

$$P(\sigma_n \leq C_1\sqrt{N}) \leq e^{-C_2 \min(p,1)N}.$$

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Discretize $\mathbb{S}^{n-1}$:

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- $\#\mathcal{N} \leq \spadesuit$;
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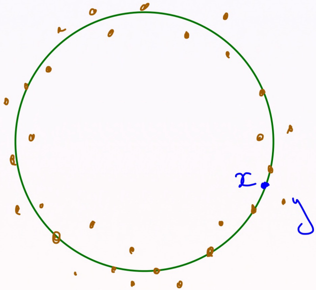
So if we know that for each y, $P(|Ay| \le 2\heartsuit) \le \frac{\diamondsuit - \clubsuit}{\spadesuit}$, we are done!

The net result

Theorem (L. 2018+) – Lite version

There exists a deterministic net $\mathcal{N} \subset \frac{3}{2}B_2^n \setminus \frac{1}{2}B_2^n$ of cardinality 1000^n such that for any integer N and any $N \times n$ random matrix A with independent columns, with probability at least $1 - e^{-5n}$, for every $x \in \mathbb{S}^{n-1}$ there exists $y \in \mathcal{N}$ such that

$$|A(x-y)| \leq \frac{100}{\sqrt{n}} \sqrt{\mathbb{E}||A||_{HS}^2}.$$



- 1000^n points
- WHP,

$|A(x-y)| \leq$ small

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$$\mathbb{S}^{n-1} \subset \cup_{x \in \mathcal{N}} \left(x + \varepsilon B_2^n\right),$$

and $\#\mathcal{N} \leq \left(\frac{3}{\varepsilon}\right)^n$

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- Then we can estimate $|A(x-y)| \le ||A||\varepsilon \le C\frac{||A||_{HS}\varepsilon}{\sqrt{n}}$?
- Recall, for any matrix A: $\frac{1}{\sqrt{n}}||A||_{HS} \le ||A|| \le ||A||_{HS}$.
- But specifically for sub-gaussian mean zero variance 1 case,

$$P\left(||A|| \ge \frac{100}{\sqrt{n}}\sqrt{\mathbb{E}||A||_{HS}^2}\right) \le e^{-5n}. \tag{1}$$

- Without strong assumptions, (1) is not true.

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Rebrova, Tikhomirov (2016) proved this Theorem assuming i.i.d. entries a_{ij}, with $\mathbb{E}a_{ij} = 0$, $\mathbb{E}a_{ij}^2 = const$, and $N = n$.

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Guedon, Litvak, Tatarko (2018) extended the result of Rebrova and Tikhomirov in the case of arbitrary n, N, and replaced i.i.d. entries with i.i.d. columns.

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Guedon, Litvak, Tatarko (2018) extended the result of Rebrova and Tikhomirov in the case of arbitrary n, N, and replaced i.i.d. entries with i.i.d. columns.

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- Advantage: the Theorem only assumes independence of columns, and no other structural assumptions!
- In particular, allowing dependent columns is crucial for the proof of the arbitrary aspect ratio result.

Proof of the net theorem – step 1: comparison via Hilbert-Schmidt

Random rounding (Alon, Klartag 2017; Klartag, L. 2018; Tikhomirov 2018;...)

Vempala, Kannan, Srinivasan,...

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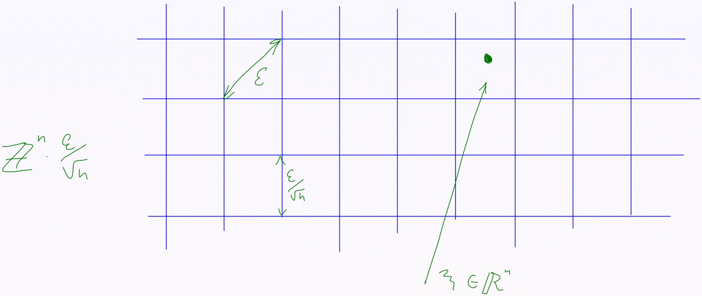

Definition

For $\xi \in \mathbb{S}^{n-1}$, write each $\xi_i = \frac{\epsilon}{\sqrt{n}}(k_i + p_i)$ for $k_i \in \mathbb{Z}$ and $p_i \in [0,1)$. Consider a random vector $\eta^\xi \in (\epsilon/\sqrt{n})\mathbb{Z}^n$:

$$\eta_i^\xi = \begin{cases} \frac{\epsilon}{\sqrt{n}} k_i, & \text{with probability } 1 - p_i \\ \frac{\epsilon}{\sqrt{n}}(k_i + 1), & \text{with probability } p_i. \end{cases}$$

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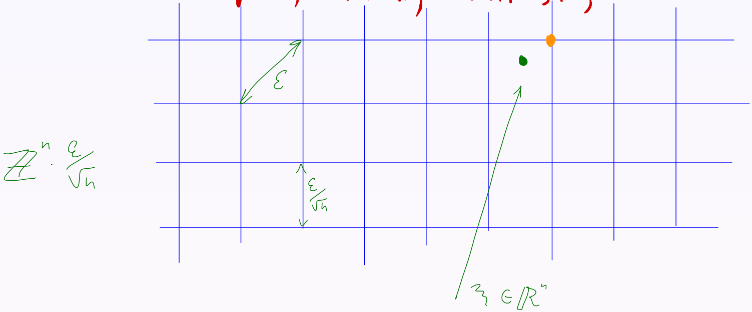

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15/33

Proof – step 1: comparison via Hilbert-Schmidt

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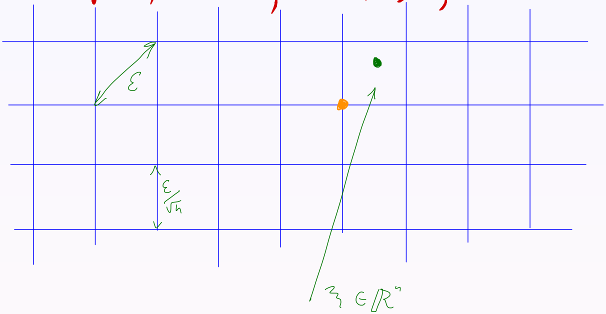
$\mathbb{Z}^n \cdot \frac{\varepsilon}{\sqrt{n}}$



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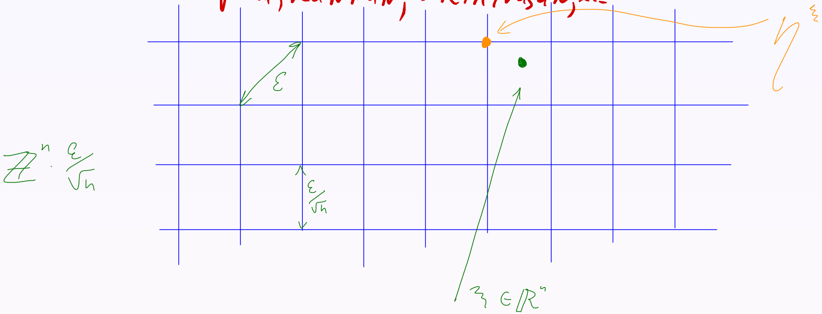

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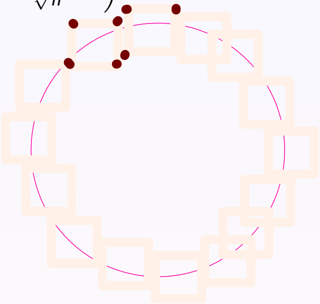

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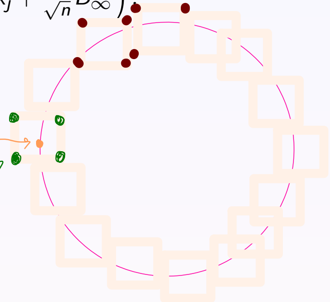
possible values of η^ξ

η



- Therefore, there is a set $\mathcal{N}$ such that for all $\xi \in \mathbb{S}^{n-1}$, we have $\eta^\xi \in \mathcal{N}$, and $\#\mathcal{N} \le \left(\frac{100}{\epsilon}\right)^n$;

Proof – step 1: comparison via Hilbert-Schmidt

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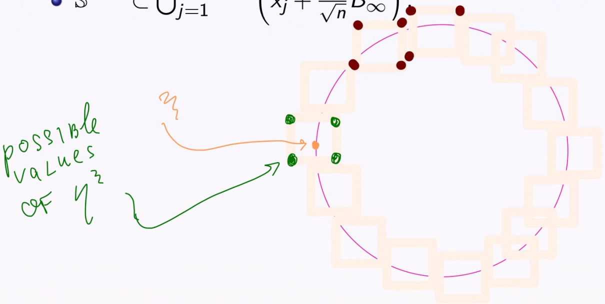

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- We have $\|\xi - \eta^\xi\|_\infty \leq \frac{\epsilon}{\sqrt{n}}$ and $\mathbb{E}\eta^\xi = \xi$;
- Hence, using the fact that $\mathbb{E}(\eta^\xi - \xi) = 0$, we get:

$$\mathbb{E}|\langle \eta^\xi - \xi, \theta \rangle|^2 \leq \frac{\epsilon^2 |\theta|^2}{n} \quad (\heartsuit)$$

Proof – step 1: comparison via Hilbert-Schmidt

Lemma 1 (comparison via Hilbert-Schmidt)

There exists a collection of points $\mathcal{F}$ with $\#\mathcal{F} \leq (\frac{C}{\epsilon})^{n-1}$ such that for any (deterministic) matrix $A : \mathbb{R}^n \to \mathbb{R}^N$, for every $\xi \in \mathbb{S}^{n-1}$ there exists an $\eta \in \mathcal{F}$ satisfying

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Proof.

- Recall: $|Ax|^2 = \sum_{i=1}^N \langle A^T e_i, x\rangle^2$, where $A^T e_i$ are the rows of A;
- By ($\heartsuit$), $\mathbb{E}_\eta |\langle \eta^\xi - \xi, A^T e_i\rangle|^2 \leq C \frac{\epsilon^2 |A^T e_i|^2}{n}$;
- Summing up, we get

$$\mathbb{E}_\eta |A(\eta^\xi - \xi)|^2 = \mathbb{E}_\eta \sum_{i=1}^N \langle A^T e_i, \eta^\xi - \xi\rangle^2 \leq \left(C' \frac{\epsilon}{\sqrt{n}} ||A||_{HS} \right)^2;$$

- If $P(\textit{find a red ball in a box}) \geq 0.1$ then there exists a red ball in a box.

$\square$

Proof – step 2: parallelepipeds

Remark

$$P(||A||_{HS}^2 \geq 10\mathbb{E}||A||_{HS}^2) \leq 0.1.$$

Thus Lemma 1 implies the Theorem with probability 0.9 rather than $1 - e^{-5n}$.
Not good:(

Remark

$$P(||A||^2_{HS} \geq 10\mathbb{E}||A||^2_{HS}) \leq 0.1.$$

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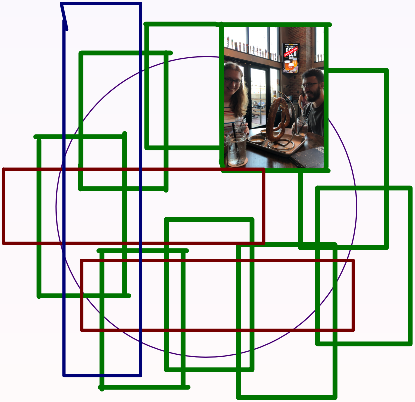
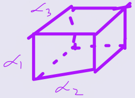
Idea of Rebrova and Tikhomirov, 2016: cover with parallelepipeds and not just cubes!



Proof – step 2: parallelepipeds

Admissible set of parallelepipeds

- For $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$ with $\alpha_i > 0$, we fix the parallelepiped



$$P_\alpha = \{x \in \mathbb{R}^n : |x_i| \leq \alpha_i\}.$$

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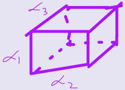

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- For $\kappa > 1$, denote $\Omega_\kappa = \left\{\alpha \in \mathbb{R}^n : \alpha_i \in [0,1], \prod_{i=1}^n \alpha_i > \kappa^{-n}\right\}$.

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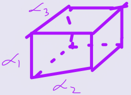

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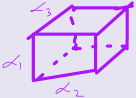

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Lemma 2 (comparison via parallelepipeds)

Pick any $\alpha \in \Omega_\kappa$. Let A be any $N \times n$ matrix. There exists a net $\mathcal{F}_\alpha$ with $\#\mathcal{F}_\alpha \le \left(\frac{100\kappa}{\epsilon}\right)^n$ such that for every $\xi \in \mathbb{S}^{n-1}$ there exists an $\eta \in \mathcal{F}_\alpha$ satisfying

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Proof – step 3: $\mathcal{B}_\kappa$ and nets on nets

Key definition: for any matrix A

$$\mathcal{B}_\kappa(A) := \min_{\alpha_i \in [0,1],\, \prod_{i=1}^n \alpha_i \geq \kappa^{-n}} \sum_{i=1}^{n} \alpha_i^2 |Ae_i|^2.$$

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Corollary of Lemma 2

Let A be any $N \times n$ matrix. There exists a small enough net $\mathcal{F}$ such that for every $\xi \in \mathbb{S}^{n-1}$ there exists an $\eta \in \mathcal{F}$ satisfying

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But the net depends on the matrix! Not good:(

Way out: discretize the admissible set Ω_κ.

$$\Omega_\kappa = \left\{\alpha : \alpha_i \in [0,1],\ \prod_{i=1}^{n} \alpha_i \geq \kappa^{-n}\right\}$$

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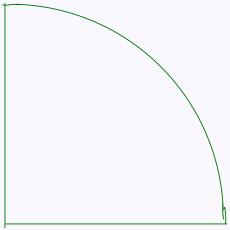

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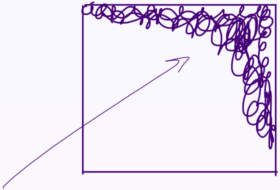

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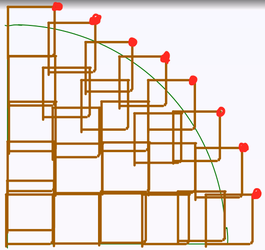

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The "nets on nets" Lemma

There exists a collection $\mathcal{F} \subset \Omega_{\kappa^2}$ of cardinality 30^n such that for any $\alpha \in \Omega_\kappa$ there exists a $\beta \in \mathcal{F}$ so that for all $i = 1, ..., n$ we have $\alpha_i^2 \geq \beta_i^2$.
In particular, for any $N \times n$ matrix A, we have

$$\mathcal{B}_\kappa(A) \geq \min_{\beta \in \mathcal{F}} \sum_{i=1}^n \beta_i^2 |Ae_i|^2.$$

A net for deterministic matrices: combining steps 1-3.

Theorem about deterministic matrices

There exists a deterministic net $\mathcal{N}$ of cardinality 1000^n such that for any integer N and any $N \times n$ deterministic matrix A, for every $x \in \mathbb{S}^{n-1}$ there exists $y \in \mathcal{N}$ such that

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This reduces the proof of the Theorem to estimating the large deviation of $\mathcal{B}_\kappa(A)$ when A is a random matrix coming from an appropriate model.

Step 4: Large deviation of $\mathcal{B}_\kappa$.

Lemma

Let A be a random matrix with independent columns. Pick any $\kappa > 1$. Then

$$P\left(B_\kappa(A) \geq 10\mathbb{E}||A||_{HS}^2\right) \leq (C\kappa)^{-2n}.$$

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Proof.

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- Consider a collection of random variables $\alpha_i^2 = \min\left(1, \frac{\mathbb{E}Y_i^2}{Y_i^2}\right)$.

Step 4: Large deviation of $\mathcal{B}_\kappa$.

Proof.

- We estimate

$$P\left(B_\kappa(A) \geq 10\mathbb{E}||A||_{HS}^2\right) \leq$$

$$P\left(\sum_{i=1}^{n} \min\left(1, \frac{\mathbb{E}Y_i^2}{Y_i^2}\right) Y_i^2 \geq 10\mathbb{E}||A||_{HS}^2\right) +$$

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- $P_1 = 0$.
- By Markov's inequality, $P_2 \leq (C\kappa)^{-2n}$.

$\square$

Summary: the non-lite version

Theorem (NON-lite)

Fix $n \in I\!N$. Consider any $S \subset \mathbb{R}^n$. Pick any $\gamma \in (1, \sqrt{n})$, $\epsilon \in (0, \frac{1}{20\gamma})$, $\kappa > 1$, $p > 0$ and $s > 0$.
There exists a (deterministic) net $\mathcal{N} \subset S + 4\epsilon\gamma B_2^n$, with

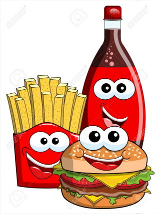
$$\#\mathcal{N} \leq \begin{cases} N(S, \epsilon B_2^n) \cdot (C_1\gamma)^{\frac{C_2 n}{\gamma^{0.08}}}, & \text{if } \log\kappa \leq \frac{\log 2}{\gamma^{0.09}}, \\ N(S, \epsilon B_2^n) \cdot (C\kappa \log\kappa)^n, & \text{if } \log\kappa \geq \frac{\log 2}{\gamma^{0.09}}, \end{cases}$$

such that for every $N \in I\!N$ and every random $N \times n$ matrix A with independent columns, with probability at least

$$1 - \kappa^{-2pn}\left(1 + \frac{1}{s^p}\right)^n,$$

for every $x \in S$ there exists $y \in \mathcal{N}$ such that

$$|A(x-y)| \leq C_3 \frac{\epsilon\gamma\sqrt{s}}{\sqrt{n}} \sqrt{\sum_{i=1}^{n} (\mathbb{E}|Ae_i|^{2p})^{\frac{1}{p}}}.$$



Here C, C_1, C_2, C_3 are absolute constants.

Summary: the non-lite version

Theorem (NON-lite)

Fix $n \in \mathbb{N}$. Consider any $S \subset \mathbb{R}^n$. Pick any $\gamma \in (1, \sqrt{n})$, $\epsilon \in (0, \frac{1}{20\gamma})$, $\kappa > 1$, $p > 0$ and $s > 0$.

There exists a (deterministic) net $\mathcal{N} \subset S + 4\epsilon\gamma B_2^n$, with

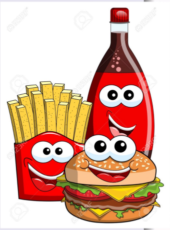
$$\#\mathcal{N} \le \begin{cases} N(S, \epsilon B_2^n) \cdot (C_1\gamma)^{\frac{C_2 n}{\gamma^{0.08}}}, & \text{if } \log\kappa \le \frac{\log 2}{\gamma^{0.09}}, \\ N(S, \epsilon B_2^n) \cdot (C\kappa \log\kappa)^n, & \text{if } \log\kappa \ge \frac{\log 2}{\gamma^{0.09}}, \end{cases}$$

such that for every $N \in \mathbb{N}$ and every random $N \times n$ matrix A with independent columns, with probability at least

$$1 - \kappa^{-2pn}\left(1 + \frac{1}{s^p}\right)^n,$$

for every $x \in S$ there exists $y \in \mathcal{N}$ such that

$$|A(x-y)| \le C_3 \frac{\epsilon\gamma\sqrt{s}}{\sqrt{n}} \sqrt{\sum_{i=1}^{n} (\mathbb{E}|Ae_i|^{2p})^{\frac{1}{p}}}.$$



Here C, C_1, C_2, C_3 are absolute constants. γ is the "sparsity" parameter

The Rudelson-Vershynin scheme: idea 1

Point-wise small ball for $|Ax|$

Fix $x \in \mathbb{S}^{n-1}$. If a_{ij} are independent and UAC, there exist constants C_1 and C_2 such that $P(|Ax| \leq C_1\sqrt{n}) \leq e^{-C_2 n}$.

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- If $S = \mathbb{S}^{n-1}$ then $\mathcal{N} = (100/C_1)^n$, and possibly $\#\mathcal{N} >> e^{C_2 n}$!
- But for some **small** set S we could get $\#\mathcal{N} \leq e^{0.5 C_2 n}$...

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A distance estimate

- $|Ax| = \sup_{u \in \mathbb{S}^{n-1}} \langle Ax, u \rangle = \sup_{u \in \mathbb{S}^{n-1}} \sum_{i=1}^{n} x_i \langle Ae_i, u \rangle$

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Conclusion: invertibility via distance estimate

Let $T \subset \mathbb{S}^{n-1}$ be such a set of x that for 99% of coordinates, $|x_i| \geq \frac{1}{10\sqrt{n}}$.

$$P\left(\inf_{x \in T} |Ax| < \frac{\varepsilon}{\sqrt{n}}\right) \leq \frac{C}{n} \sum_{i=1}^{n} P\left(dist(Ae_i, H_i) < \epsilon\right).$$

The Rudelson-Vershynin scheme: combining idea 1 and idea 2

Decomposition of the sphere

$$\mathbb{S}^{n-1} = Comp_{\delta,\rho} \cup Incomp_{\delta,\rho},$$

where

$$Comp_{\delta,\rho} = \{x \in \mathbb{S}^{n-1} : \{i : |x_i| \geq \frac{\rho}{\sqrt{n}}\} \leq \delta n\};$$

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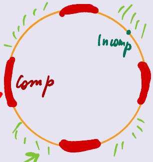

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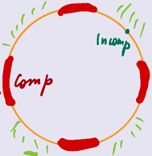

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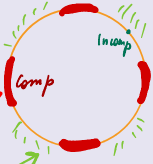


- The set $Comp_{\delta,\rho}$ has small entropy and one may apply the net argument;
- The set $Incomp_{\delta,\rho}$ can be handled by the distance estimate, provided that we can prove the small ball estimate for $dist(Ae_i, H_i)$.

The distance theorem

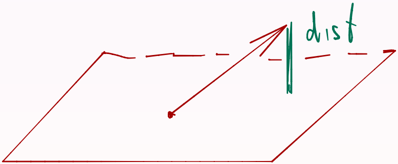
Theorem about distances (L, Tikhomirov, Vershynin, 2019+)

Let A have independent UAC entries and $\mathbb{E}||A||_{HS}^2 \leq Kn^2$. Denote

$$H_j = span\{Ae_i : i \neq j, i = 1, \dots, n\};$$

Take any $j \leq n$ such that $\mathbb{E}|Ae_j|^2 \leq rn^2$. Then

$$P\left(dist(Ae_j, H_j) \leq \varepsilon\right) \leq C\varepsilon + 2e^{-cn}, \quad \varepsilon \geq 0.$$



Sketch of the proof of the distance theorem

Esseen's Lemma

Given a variable ξ with the characteristic function $\varphi(\cdot) = \mathbb{E}\exp(2\pi \mathbf{i}\xi\cdot)$,

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RLCD – definition

For a random vector X in $\mathbb{R}^n$, a (deterministic) vector v in $\mathbb{R}^n$, and parameters $L > 0$, $u \in (0,1)$, define

$$RLCD^X_{L,u}(v) := \inf\left\{\theta > 0 : \mathbb{E}dist^2(\theta v \star \overline{X}, \mathbb{Z}^n) < \min(u|\theta v|^2, L^2)\right\}.$$

Here by $\star$ we denote the Schur product

$$v \star X := (v_1 X_1, \ldots, v_n X_n).$$

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Note: Rudelson-Vershynin previously defined LCD, a parameter which worked well to study the i.i.d. case.

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Geometrical meaning of RLCD

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Anticoncentration via RLCD

Let $X = (X_1, \ldots, X_n)$ be a random vector with independent coordinates satisfying $\max_i P(\sup_{z\in\mathbb{R}} |X_i - z| < 1) \le b$ for some $b \in (0,1)$. Let $c_0 > 0$, $L > 0$ and $u \in (0,1)$. Then for any vector $v \in \mathbb{R}^n$ with $|v| \ge c_0$ and any $\varepsilon \ge 0$, we have

$$P(\langle X, v\rangle < \varepsilon) \le C\varepsilon + C\exp(-\widetilde{c}L^2) + \frac{C}{RLCD^X_{L,u}(v)}.$$

Here $C > 0, \widetilde{c} > 0$ may only depend on b, c_0, u.

In words

If RLCD of a vector v is large, then the scalar product $\langle X, v\rangle$ has great anti-concentration properties!

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- Combining these bounds allows to iterate on the level sets and to obtain the distance theorem.

Thanks for your attention!