Bounding marginal densities of product measures.

(based on the joint work with Grigoris Paouris and Peter Pivovarov)

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MFO,
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Marginal density

If f is a probability density on $\mathbb{R}^n$ and E is a subspace, the marginal density of f on E is defined by

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Example: consider $K \subset \mathbb{R}^n$ such that $|K| = 1$. Let $f = 1_K$. Then

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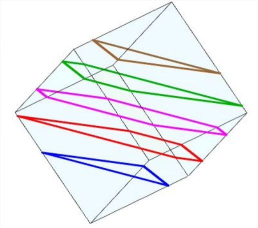

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Hence it is of interest to bound $||\pi_E(f)(x)||_\infty$ from above and the bound should ideally look like C^k. A number of related studies were conducted by: Ball, Barthe, Bobkov, Brzezinski, Chistyakov, Dann, Gluskin, Koldobsky, König, Paouris, Pivovarov, Rogozin, Rudelson, Vershynin,...

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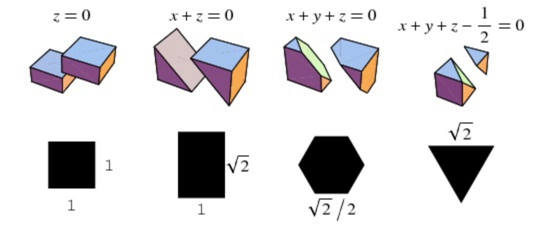

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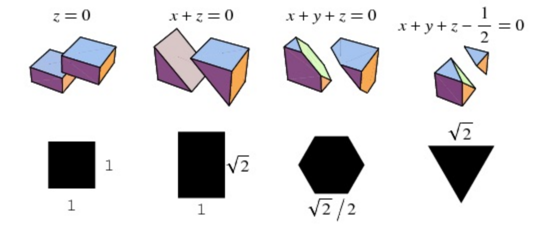

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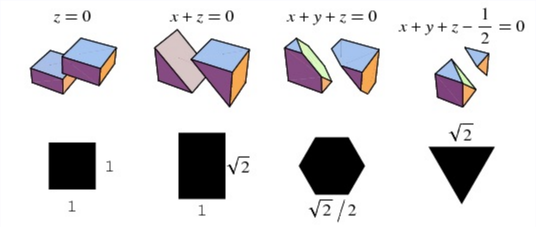

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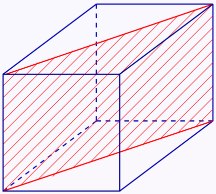
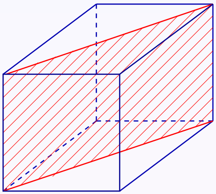
This estimate is sharp!



Ball's Theorems about unit cube



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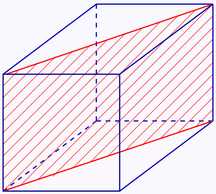

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Fix $k \in [1, n]$. For every subspace H of codimension k,

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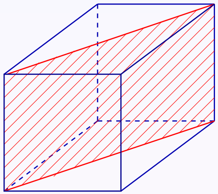

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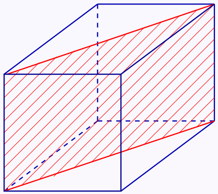

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- The estimate $\left(\frac{n}{n-k}\right)^{\frac{n-k}{2}} \leq \sqrt{e}^k$ is sharp for the case $n-k \mid k$.

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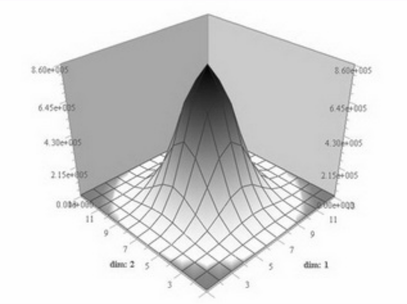

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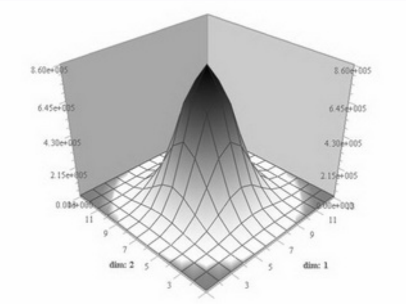

Can $C = \sqrt{2}$ like in the case of the unit cube?

Main result

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Corollary

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- Idea of the proof: Layers of product measures with symmetric decreasing components are coordinate boxes. We shall estimate sections of coordinate boxes using Ball's techniques for the cube and the layer-cake formula.

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Ball-like propositions

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Proposition 2

Let $1 \leq k \leq n/2$ and $H \in G_{n,n-k}$. Then there exists $\{\beta_j\}_{j=1}^n \subset [0,1]$ with $\sum_{i=1}^n \beta_i = n-k$ such that for any $z_1, \dots, z_n \in \mathbb{R}^+$, the box $B = \prod_{j=1}^n [-z_j/2, z_j/2]$ satisfies

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$$\begin{aligned}\pi_E(f)(0) &= \int_{\mathbb{R}^{n-k}} \prod_{i=1}^{n} f_i(\langle y, w_i\rangle) dy \\ &\leq \int_{\mathbb{R}^{n-k}} \prod_{i=1}^{n} f_i^*(\langle y, w_i\rangle) dy \\ &= \int_0^{c_1} \cdots \int_0^{c_n} \int_{\mathbb{R}^{n-k}} \prod_{i=1}^{n} 1_{\{f_i^* > t_i\}}(\langle x, w_i\rangle) dx dt_1 \dots dt_n\end{aligned}$$

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&= \int_0^{c_1} \cdots \int_0^{c_n} |B(t_1, ..., t_n) \cap E^{\perp}| dt_1 \dots dt_n.
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- CASE 2: there exists a unit vector $b \in H^\perp$ and $i \in [1, n]$ such that $b_i \geq \frac{1}{\sqrt{2}}$.

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Note: $a_i \leq \frac{1}{\sqrt{2}}$.

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By Fourier inversion formula,

$$|B \cap H| = \pi_{H^{\perp}}(1_B)(0) = \frac{1}{(2\pi)^k} \int_{H^{\perp}} \Phi(w) dw.$$

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&= \prod_{j=1}^{n} \frac{2 \sin \frac{1}{2} z_j a_j \langle w, u_j \rangle}{z_j a_j \langle w, u_j \rangle}.
\end{aligned}$$

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Upshot

$$|B \cap H| \leq \frac{1}{\pi^k} \int_{H^{\perp}} \prod_{j=1}^{n} \Phi_j(\langle w, u_j \rangle) dw,$$

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Theorem 1 (Ball)

Let $u_1, \ldots, u_n$ be unit vectors in $\mathbb{R}^k$, $k \le n$, and $c_1, \ldots, c_n > 0$ satisfying $\sum_1^n c_i u_i \otimes u_i = I_k$. Then for integrable functions $f_1, \ldots, f_n : \mathbb{R} \to [0, \infty)$,

$$\int_{\mathbb{R}^k} \prod_{i=1}^{n} f_i(\langle u_i, x \rangle)^{c_i} dx \le \prod_{i=1}^{n} \left(\int_{\mathbb{R}} f_i \right)^{c_i}.$$

There is equality if the f_i's are identical Gaussian densities.

The above Theorem of Ball is used with $c_j = \frac{1}{a_j^2}$:

$$|B \cap H| \leq \frac{1}{\pi^k} \prod_{j=1}^{n} \left(\int_{\mathbb{R}} \Phi_j(t)^{\frac{1}{a_j^2}} \, dt \right)^{a_j^2}$$

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For every $p \geq 2$,

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \left| \frac{\sin t}{t} \right|^p dt \leq \sqrt{\frac{2}{p}}.$$

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• Application of the above Theorem with $p = \frac{1}{a_j^2}$ and rescaling finish the proof in the case 1.

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$$|B \cap H| \leq \frac{1}{b_i} |\widetilde{B} \cap \widetilde{H}|,$$

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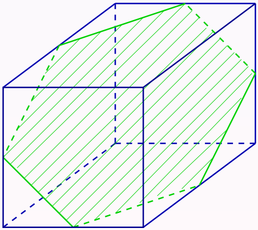

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- If $k \geq 2$, we use induction. $\square$

Thanks for your attention!