On the geometry of log-concave measures.

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May 4, 2015.

Chapters 2 and 3: Maximal Surface Area

Classical Isoperimetric Inequality

Isoperimetric Inequality

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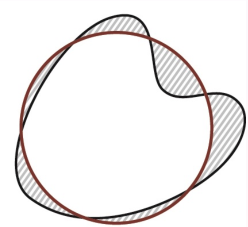

But if we replace the usual Lebesgue volume measure with another measure, the answer to that question may change!

Gaussian isoperimetric type inequalities

Gaussian Measure

The Standard Gaussian Measure γ_2 on $\mathbb{R}^n$ is the probability measure with density

$$\varphi_2(y) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{|y|^2}{2}}$$

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There is a convenient integral expression for $\gamma_2(\partial Q)$:

$$\gamma_2(\partial Q) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{\partial Q} e^{-\frac{|y|^2}{2}} d\sigma(y),$$

where $d\sigma(y)$ stands for Lebesgue surface measure.

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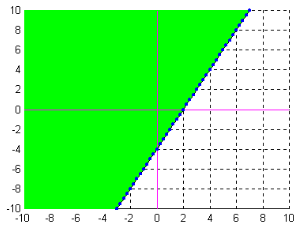

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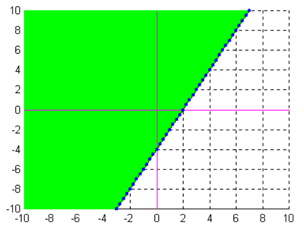

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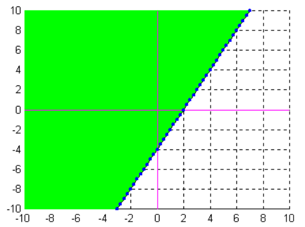

How to ask the reverse question?

By $\mathcal{K}_n$ we denote the set of all convex bodies in $\mathbb{R}^n$.
Let Q run over $\mathcal{K}_n$. What is the maximal Gaussian surface area of Q?

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Are there any other interesting measures for which it is natural to ask for Isoperimetric type inequalities?

Log concave measures

Definition of log concave measures

A Borel measure μ on $\mathbb{R}^n$ is called log concave, if for any compact sets $A, B \subset \mathbb{R}^n$ and for any $\lambda \in [0,1]$,

$$\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^{\lambda} \cdot \mu(B)^{1-\lambda}.$$

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Question (generalization of Ball-Nazarov Theorems)

Fix a log concave rotation invariant measure γ on $\mathbb{R}^n$ with density $C_n e^{-\varphi(|y|)}$ on $\mathbb{R}^n$. Let Q be a convex body in $\mathbb{R}^n$. What are the bounds for $\max \gamma(\partial Q)$?

Rotation invariant Log concave measures

First example to try: let $p > 0$, consider probability measure γ_p on $\mathbb{R}^n$ with density

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Theorem (G. L., JMAA 2013)

For any positive p

$$c(p) n^{\frac{3}{4}-\frac{1}{p}} \leq \max \gamma_p(\partial Q) \leq C(p) n^{\frac{3}{4}-\frac{1}{p}},$$

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For $p \geq 1$ the measure γ_p is log concave, but for $p < 1$ it is not.

The reverse isoperimetric inequality for Rotation invariant Log concave measures. The main result.

Theorem (G. L., GAFA seminar notes, 2014)

Fix $n \geq 2$. Let γ be log concave rotation invariant measure on $\mathbb{R}^n$. Consider a random vector X in $\mathbb{R}^n$ distributed with respect to γ.

$$\max_{Q \in \mathcal{K}_n} \gamma(\partial Q) = C \frac{\sqrt{n}}{\sqrt{\mathbb{E}|X|} \sqrt[4]{Var|X|}},$$

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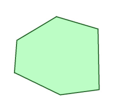
- The above is also applicable for normalized Lebesgue measure restricted on a unit ball: if X is distributed uniformly in the unit ball, $\mathbb{E}|X| \approx 1$ and $Var|X| \approx \frac{1}{n^2}$. The maximum for the surface area is attained on the unit sphere and is of order n.

Chapter 4: Gaussian surface area of a polytope with K faces

Gaussian surface area of a polytope with K faces

Let $P=\cap_{i=1}^{K}\{\langle x,\theta_i\rangle \leq \rho_i\}$ be a polytope with at most K faces.



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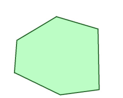

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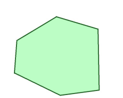

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What about log-concave rotation invariant case?

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Fix $n \geq 2$. *Let* γ *be log concave rotation invariant measure on* $\mathbb{R}^n$. *Consider a random vector* X *in* $\mathbb{R}^n$ *distributed with respect to* γ.

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Theorem (G.L., 2014)

Fix $n \geq 2$. Let γ be log concave rotation invariant measure on $\mathbb{R}^n$. Consider a random vector X in $\mathbb{R}^n$ distributed with respect to γ. Fix positive integer $K \in [2, \exp\left(\sqrt{\frac{c\mathbb{E}|X|}{\sqrt{Var|X|}}}\right)]$. Let P be a convex polytope in $\mathbb{R}^n$ with at most K facets.

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Corollary

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where $\mathbb{E}|X|$ and $Var|X|$ denote the expectation and the variance of X correspondingly, and C and c stand for absolute constants.

• In particular, this Theorem shows that the result of Nazarov for the Gaussian case is exact.

Chapter 5: On the Gaussian concentration

On the Gaussian concentration

For a measurable set $Q \subset \mathbb{R}^n$ we define a function

$$\alpha_Q(h) : \mathbb{R}^+ \to \mathbb{R}$$

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$$\alpha_Q(h) := 1 - \gamma_2(Q + hB_2^n).$$

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Moreover,

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Theorem (G.L., 2014)

$$\alpha_Q(h) \leq 1 - \gamma_2(Q) - \frac{\sqrt{\pi}\gamma_2(\partial Q)^2}{8\sqrt{n}} \cdot \left(1 - e^{-\frac{\sqrt{n}}{\sqrt{\pi}\gamma_2(\partial Q)} h}\right).$$

Chapter 6: the Gaussian Brunn-Minkowski inequality

Classical Brunn-Minkowski inequality

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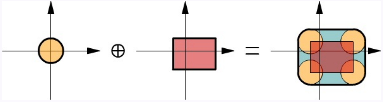
Recall: the Minkowski sum of the sets K and Q in $\mathbb{R}^n$ is the set

$$K + Q = \{a + b \,|\, a \in K, b \in Q\}.$$



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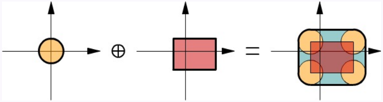


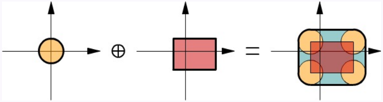
The $\lambda-$dilate of a set A in $\mathbb{R}^n$ is the set

$$\lambda A := \{\lambda a \,|\, a \in A\}.$$

Classical Brunn-Minkowski inequality

Recall: the Minkowski sum of the sets K and Q in $\mathbb{R}^n$ is the set

$$K + Q = \{a + b \,|\, a \in K, b \in Q\}.$$



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Brunn-Minkowski inequality

The classical Brunn-Minkowski inequality states that for any measurable sets $A, B \subset \mathbb{R}^n$ and for any $\lambda \in [0,1]$,

$$|\lambda A + (1-\lambda)B|^{\frac{1}{n}} \geq \lambda |A|^{\frac{1}{n}} + (1-\lambda)|B|^{\frac{1}{n}},$$

where $|\cdot|$ stands for the Lebesgue Measure on $\mathbb{R}^n$.

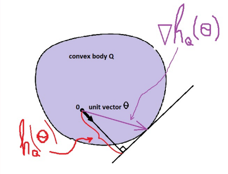
The Brunn-Minkowski inequality and "shadow systems" (highlights of the work done by Colesanti)

The support function of a convex set

Recall, that the support function h_Q of a convex set $Q \subset \mathbb{R}^2$ is the function on the unit sphere defined by

$$h_Q(\theta) = \max_{x \in Q} \langle x, \theta \rangle.$$

By homogeneity it extends from the sphere to the whole space. The support function represents the distance from the origin to the support hyperplane of a convex set in a given direction:



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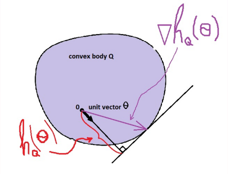

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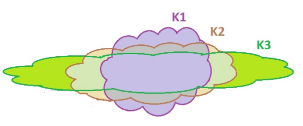

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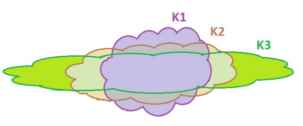

1 A. Colesanti, *From the Brunn-Minkowski inequality to a class of Poincare' type inequalities,* Communications in Contemporary Mathematics, 10 n. 5 (2008), 765-772.

Highlights of the work done by Colesanti

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Pick convex sets A and B in $\mathbb{R}^2$ with the support functions $h_A(u)$ and $h_B(u)$. For $s \in [0,1]$, consider the "support function shadow system" $\{K_s\}$, where the support function of K_s is $h_s = h_A + s(h_B - h_A)$. This way, $K_0 = A$ and $K_1 = B$.

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Claim

The Brunn-Minkowski inequality holds true for every pair of convex sets in $\mathbb{R}^2$ if and only if for every convex smooth function $h(u)$ on $\mathbb{S}^1$ and for every smooth function $\psi(u)$ on $\mathbb{S}^1$,

$$f''(0) = \left(|K_s|^{\frac{1}{2}}\right)'' |_{s=0} \leq 0.$$

The Gaussian Brunn-Minkowski inequality

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Recall: the standard Gaussian Measure γ_2 on $\mathbb{R}^n$ is the measure with density

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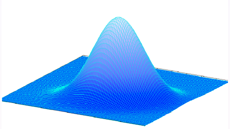

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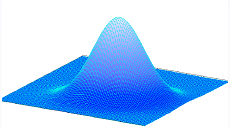

Gaussian Brunn-Minkowski inequality

Gardner and Zvavitch conjectured that for the standard Gaussian measure γ_2 the inequality analogous to BM holds *under some natural assumptions* on the sets A and B in $\mathbb{R}^n$:

$$\gamma_2(\lambda A+(1-\lambda)B)^{\frac{1}{n}} \geq \lambda\gamma_2(A)^{\frac{1}{n}}+(1-\lambda)\gamma_2(B)^{\frac{1}{n}}.$$

The Gaussian Brunn-Minkowski inequality

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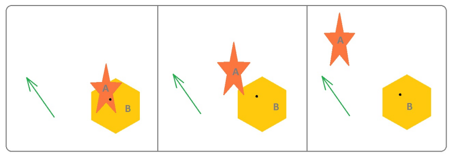
The inequality (3) is false in the full generality: one may shift the set A away from the origin. The farther the shift, the smaller the right hand side of (3) becomes, while the left hand side stays bounded from below by the fixed quantity $(1-\lambda)\gamma_2(B)^{\frac{1}{n}}$.



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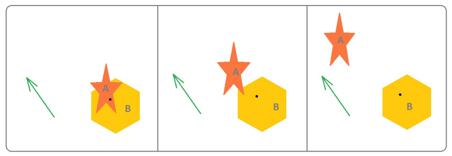

That gives a clue on which assumptions must be reinforced.

Gaussian Brunn Minkowski inequality: questions

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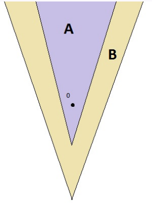

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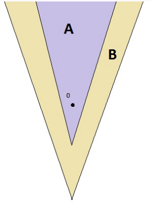

Question 2

Gardner, Zvavitch, and Nayar and Tkozh conjectured: *The Gaussian Brunn-Minkowski inequality holds true for all* ***symmetric convex sets*** *A* ***and*** *B****.***

The approach

Once again, a support function shadow system

Pick a positive number a. Let $h(u)$ be a strictly convex C^2-smooth function on the circle $\mathbb{S}^1$. Consider a function $\psi(u) \in C^2(\mathbb{S}^1)$. Let $s \in [0, a]$. Consider a family of sets K_s in $\mathbb{R}^2$, where the support function of each K_s is $h_s = h + s\psi$.

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Formula for the Gaussian measure via the support function

Let γ_2 be the Standard Gaussian measure in $\mathbb{R}^2$. Let K be a strictly convex body in $\mathbb{R}^2$ containing the origin with the support function $h(u) \in C^2(\mathbb{S}^1)$. Then

$$\gamma_2(K) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h^2 + h\ddot{h}}{h^2 + \dot{h}^2} \left(1 - e^{-\frac{h^2 + \dot{h}^2}{2}}\right) du.$$

Sketch of the proof of the formula

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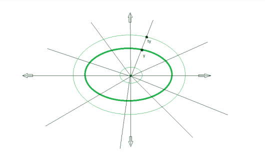

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$$\gamma_2(K) = \frac{1}{2\pi} \int_{\partial K} \int_0^1 t|y| \cos(y, n_y) e^{-\frac{(t|y|)^2}{2}} dt d\sigma(y)$$

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- In the latter integral we make the change of variables via Gauss map, passing the integration from ∂K to $\mathbb{S}^1$. The Jacobian of the Gauss map is the curvature function of K, which in the planar case is $h + \ddot{h}$, where h is the support function of K.

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- Observation that $|\nabla h|^2 = h^2 + \dot{h}^2$, and integration in t leads to the desired conclusion

$$\gamma_2(K) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h^2 + h\ddot{h}}{h^2 + \dot{h}^2} \left(1 - e^{-\frac{h^2 + \dot{h}^2}{2}}\right) du. \square$$

The general statement

The formula for any measure in $\mathbb{R}^n$

Let γ be a measure in $\mathbb{R}^n$ with density $f(x)$. Let K be a strictly convex body in $\mathbb{R}^n$ containing the origin with the support function $h(u) \in C^2(\mathbb{S}^{n-1})$, where $u \in \mathbb{S}^{n-1}$. Let $\det Q(h(u))$ be the curvature function of K. Denote the gradient of h by ∇h. Then

$$\gamma(K) = \int_{\mathbb{S}^{n-1}} \frac{h(u) \det Q(h(u))}{|\nabla h(u)|^n} \int_0^{|\nabla h|} t^{n-1} f(t \cdot \frac{\nabla h}{|\nabla h|}) dt du.$$

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This formula might find its use in other questions, such as B-Theorem, S-Theorem, Isoperimetric inequalities etc.

The neighborhood of the disc

Once again, a shadow system for $h(u) = R$

Pick a positive number a. Pick a positive number R. Consider a function $\psi(u) \in C^2(\mathbb{S}^1)$. Let $s \in [0, a]$. Consider a family of sets K_s in $\mathbb{R}^2$, where the support function of each K_s is $h_s = R + s\psi$.

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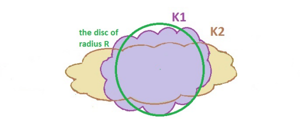

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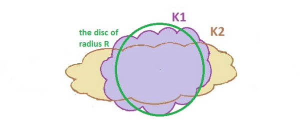

Gaussian Brunn-Minkovski is true in a neighborhood of any disc

Pick $R \in (0, \infty)$. Fix $\psi \in C^2(\mathbb{S}^1)$. Then there exists an $\epsilon = \epsilon(R, \psi)$ such that for every $K, L \in \mathbf{K}_2(R, \psi, \epsilon)$ and for every $\lambda \in [0, 1]$,

$$\gamma_2^{\frac{1}{2}}(\lambda K + (1-\lambda)L) \geq \lambda \gamma_2^{\frac{1}{2}}(K) + (1-\lambda)\gamma_2^{\frac{1}{2}}(L).$$

Sketch of the proof

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- We apply the formula for the Gaussian measure

$$\gamma_2(K) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{h^2 + h\ddot{h}}{h^2+\dot{h}^2}\left(1 - e^{-\frac{h^2+\dot{h}^2}{2}}\right) du$$

when the support function of the set is $h = R + s\psi$

$$\gamma(s) := \int_{-\pi}^{\pi} \frac{(R+s\psi)^2 + (R+s\psi)s\ddot{\psi}}{(R+s\psi)^2 + (s\dot{\psi})^2}(1 - e^{-\frac{(R+s\psi)^2+(s\dot{\psi})^2}{2}})du.$$

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$$\gamma_2(K) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h^2 + h\ddot{h}}{h^2 + \dot{h}^2} \left(1 - e^{-\frac{h^2+\dot{h}^2}{2}}\right) du$$

when the support function of the set is $h = R + s\psi$

$$\gamma(s) := \int_{-\pi}^{\pi} \frac{(R+s\psi)^2 + (R+s\psi)s\ddot{\psi}}{(R+s\psi)^2 + (s\dot{\psi})^2} (1 - e^{-\frac{(R+s\psi)^2+(s\dot{\psi})^2}{2}}) du.$$

- We differentiate it at zero twice.

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- We differentiate it at zero twice. We observe that $(\sqrt{\gamma(s)})''_0 \leq 0$ whenever

$$2(e^{\frac{R^2}{2}} - 1)\int \left[(1-R^2)\psi^2 - \dot{\psi}^2\right] - R^2\left(\int \psi\right)^2 \leq 0.$$

Sketch of the proof

We want to prove that

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- We arrive to an inequality

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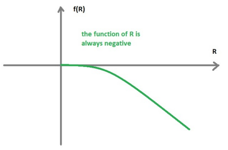
for $R>0$, which we brutal force.□



Thanks for your attention!