An inequality for the normal derivative of the Lane–Emden ground state

Simon Larson
Chalmers University of Technology &
The University of Gothenburg

joint work with

Rupert L. Frank, LMU Munich & Caltech

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Setting up the problem

Consider the variational problem

$$\lambda_q(\Omega) := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_{L^2(\Omega)}^2}{\|u\|_{L^q(\Omega)}^2},$$

with $1 \leq q \leq 2$ and $\Omega \subset \mathbb{R}^d$ is an open set of finite measure.

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Today we are interested in **properties of minimizers** and how they depend on the set Ω.

Throughout $u_{q,\Omega}$ denotes a non-negative minimizer normalized in $L^q(\Omega)$.

Such minimizers solve the **Lane–Emden equation**

$$\begin{cases} -\Delta u_{q,\Omega} = \lambda_q(\Omega) u_{q,\Omega}^{q-1} & \text{in } \Omega\,, \\ u_{q,\Omega} = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Remarks:

- If $1 \leq q < 2$ then $u_{q,\Omega}$ is unique. If $q = 2$ and Ω has multiple connected components then there might be several (finitely many) normalized minimizers.
- For $q = 1$ the right-hand side of the equation should be understood as $\lambda_1(\Omega) u^0_{1,\Omega} \equiv \lambda_1(\Omega)$.

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By homogeneity

$$\tilde{u} = \lambda_q(\Omega)^{-1/(2-q)} u_{q,\Omega} \quad \text{and} \quad \mathbf{F}_q(\Omega) = \lambda_q(\Omega)^{-q/(2-q)}.$$

In particular, the quantity $\mathbf{F}_1(\Omega) = 1/\lambda_1(\Omega)$ is the torsional rigidity of Ω and the solution $\tilde{u}$ is the classical torsion function; $-\Delta w = 1$ with $w|_{\partial\Omega} = 0$.

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Remarks: We won't see it much today but there are interesting differences between the cases of $q = 2$ and $1 < q < 2$ (see e.g. Brasco–Franzina '20)

- If $q = 2$ then the critical values of $u \mapsto \|\nabla u\|_{L^2}^2 / \|u\|_{L^2}^2$ is an infinite discrete set (the spectrum of the Dirichlet Laplacian).
- For $1 < q < 2$ the critical values of $u \mapsto \|\nabla u\|_{L^2}^2 / \|u\|_{L^q}^2$ is a closed infinite set but it is not in general known to be countable. There are examples where the set fails to be discrete (there are examples where $\lambda_q(\Omega)$ is an accumulation point).

Basic properties (see Brasco–Franzina '20)

i) (*monotonicity*) If $\Omega' \subset \Omega$, then $\lambda_q(\Omega') \geq \lambda_q(\Omega)$.

ii) (*scaling*) Let $\alpha_q = (2 + d(2/q - 1))^{-1}$, then for all $s > 0$

$$\lambda_q(s\Omega) = s^{-1/\alpha_q}\lambda_q(\Omega) \quad \text{and} \quad u_{q,s\Omega}(x) = s^{-d/q}u_{q,\Omega}(x/s)\,.$$

iii) (*disjoint unions*) If $\Omega = \bigcup_{j\geq 1}\Omega_j$ with $\Omega_j \cap \Omega_{j'} = \emptyset$ when $j \neq j'$, then

a) for $1 \leq q < 2$

$$\lambda_q(\Omega) = \Bigl(\sum_{j\geq 1}\lambda_q(\Omega_j)^{-\frac{q}{2-q}}\Bigr)^{-\frac{2-q}{q}} \quad \text{and} \quad u_{q,\Omega} = \sum_{j\geq 1}\Bigl(\frac{\lambda_q(\Omega)}{\lambda_q(\Omega_j)}\Bigr)^{\frac{1}{2-q}}u_{q,\Omega_j}\,.$$

b) for $q = 2$

$$\lambda_2(\Omega) = \min_{j\geq 1}\lambda_2(\Omega_j)$$

and the set of minimizers is the linear span of

$$\{u_{q,\Omega_j} : j \geq 1 \text{ such that } \lambda_2(\Omega_j) = \lambda_2(\Omega)\}.$$

iv) (*continuity interior exhaustion*) If $\Omega \subset \mathbb{R}^d$ has finite measure and $\{\Omega_j\}_{j\geq 1}$ satisfy $\Omega_j \subset \Omega_{j+1}$ and $\cup_{j\geq 1}\Omega_j = \Omega$ and $\Omega_j \to \Omega$ locally in the Hausdorff distance then

$$\lim_{j\to\infty}\lambda_q(\Omega_j) = \lambda_q(\Omega)\,.$$

Main result

Theorem

Fix $1 \le q \le 2$*, let* $\Omega \subset \mathbb{R}^d$ *be open and bounded with Lipschitz boundary. Then*

$$\int_{\partial\Omega} \Big(\frac{\partial u_{q,\Omega}}{\partial \nu} \Big)^2 \, d\mathcal{H}^{d-1}(x) \ge \frac{\lambda_q(\Omega)^{1+\alpha_q}}{\alpha_q \lambda_q(B)^{\alpha_q}},$$

where B *is the unit ball and* $\alpha_q = (2 + d(2/q - 1))^{-1}$*.*

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Remarks:

- That the normal derivative $\frac{\partial u_{q,\Omega}}{\partial \nu}$ can be made sense of when $\partial\Omega$ is irregular follows from classical work of Dahlberg, Jerison–Kenig, Verchota in the 70's and 80's.
- By the Pohozaev identity $\int_{\partial\Omega} \big(\frac{\partial u_{q,\Omega}}{\partial \nu}\big)^2 x \cdot \nu \, d\mathcal{H}^{d-1}(x) = \frac{\lambda_q(\Omega)}{\alpha_q}$ equality holds if Ω is a ball.
- If Ω^* denotes a ball of the same measure as Ω, the theorem combined with the Faber–Krahn-type inequality $\lambda_q(\Omega) \geq \lambda_q(\Omega^*)$ implies that

$$\int_{\partial\Omega} \Big(\frac{\partial u_{q,\Omega}}{\partial \nu}\Big)^2 \, d\mathcal{H}^{d-1}(x) \geq \int_{\partial\Omega^*} \Big(\frac{\partial u_{q,\Omega^*}}{\partial \nu}\Big)^2 \, d\mathcal{H}^{d-1}(x)\,.$$

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History: For convex sets bounds of this form have appeared earlier, in particular in connection to Minkowski-type problems:

- For $q = 2$ the bound is (implicitly) in Jerison Adv. Math. '96 (for problem of electrostatic capacity an analogue appears in Jerison Acta Math. '96).
- For $q = 1$ the bound is (implicitly) in Colesanti–Fimiani '10.
- For $q \in \{1, 2\}$ the bounds appear in Bucur–Fragala–Lamboley '12.
- Similar results but where the Laplacian is replaced by the p-Laplace operator appear in Colesanti–Nyström–Salani–Xiao–Yang–Zhang '15.

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Idea: Our aim is to mimic a classical argument to pass from the **classical Brunn–Minkowski inequality** to the **classical isoperimetric inequality**.

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$$\mathcal{SM}_*(\Omega) := \liminf_{t\to 0^+} \frac{|\Omega + tB| - |\Omega|}{t} = \liminf_{t\to 0^+} \frac{|(\Omega + tB) \setminus \Omega|}{t}\,.$$

When can we relate this quantity to something we are (more) familiar with?

Strategy of proof

Here the strategy boils down to:

1) a Brunn–Minkowski inequality for λ_q, and
2) computing (one-sided) derivative of $t \mapsto \lambda_q(\Omega + tB)$ at $t = 0$.

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Part 1) is ok.

Theorem

For $0 \le s \le 1$ and $\Omega_0, \Omega_1 \subset \mathbb{R}^d$ open sets of finite measure

$$\lambda_q((1-s)\Omega_0 + s\Omega_1) \le \Big((1-s)\lambda_q(\Omega_0)^{-\alpha_q} + s\lambda_q(\Omega_1)^{-\alpha_q}\Big)^{-1/\alpha_q}.$$

This is (essentially) proved in
- Brascamp–Lieb '76 for $q = 2$,
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Issues:

- The dependence of λ_q on regular perturbations of Ω is rather delicate.
- Generally the set $\Omega + tB$ is not a regular perturbation of Ω for t small.

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For the rest of the talk we take a look at the first two points and aim to prove:

Lemma

Fix $1 \leq q \leq 2$, let $\Omega \subset \mathbb{R}^d$ be open, bounded, connected with C^1 boundary. Then

$$\lim_{t \to 0^+} \frac{\lambda_q(\Omega + tB) - \lambda_q(\Omega)}{t} = -\int_{\partial\Omega} \Big(\frac{\partial u_{q,\Omega}}{\partial \nu}\Big)^2 \, d\mathcal{H}^{d-1}(x)\,.$$

A Hadamard formula for $\lambda_q(\Omega)$

Theorem

Fix $1 \leq q \leq 2$ *and* $\Omega \subset \mathbb{R}^d$ *open, bounded, and connected. Let* $\Phi \in C^1((-1,1); W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d))$, *be such that* $\Phi(t,\cdot)\colon \mathbb{R}^d \to \mathbb{R}^d$ *is a bi-Lipschitz homeomorphism of a neighbourhood of* Ω *onto its image, and*

$$\Phi(t,x) = x + t\dot{\Phi}(x) + o_{t\to 0}(t) \quad \text{in } W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)\,.$$

Then $t \mapsto \lambda_q(\Phi(t,\Omega))$ *is differentiable at* $t=0$ *and*

$$\begin{aligned}\lim_{t\to 0}\frac{\lambda_q(\Phi(t,\Omega))-\lambda_q(\Omega)}{t} &= -2\int_\Omega \nabla u_{q,\Omega}\cdot \big(D\dot{\Phi}\big)\nabla u_{q,\Omega}\,dx \\ &\quad + \int_\Omega \Big(|\nabla u_{q,\Omega}|^2 - \frac{2}{q}\lambda_q(\Omega)u_{q,\Omega}^q\Big)\nabla\cdot\dot{\Phi}\,dx\,.\end{aligned}$$

If Ω *has Lipschitz boundary,*

$$\lim_{t\to 0}\frac{\lambda_q(\Phi(t,\Omega))-\lambda_q(\Omega)}{t} = -\int_{\partial\Omega}\left(\frac{\partial u_{q,\Omega}}{\partial\nu}\right)^2 \nu\cdot\dot{\Phi}\,d\mathcal{H}^{d-1}(x)\,.$$

Remark: For $q=1$ or 2 this is classical.

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$$H_0^1(\Omega) \times \mathbb{R} \times (-1,1) \to H^{-1}(\Omega) \times \mathbb{R}$$

$$\begin{pmatrix} v \\ \lambda \\ t \end{pmatrix} \mapsto \begin{pmatrix} -(\Delta(v \circ \Phi(t,\cdot)^{-1})) \circ \Phi(t,\cdot) - \lambda v^{q-1} \\ \int_\Omega |v|^q |\det D_x\Phi(t,x)|\,dx - 1 \end{pmatrix}$$

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Problem: for $1 < q < 2$ the map $v \mapsto v^{q-1}$ is **not** Fréchet differentiable.

Solution: Use a variational proof which avoids differentiating $t \mapsto u_{q,\Phi(t,\Omega)}$.

A Hadamard formula for $\lambda_q(\Omega)$

Define

$$\mathcal{F}_t \colon H^1_0(\Omega) \to \mathbb{R}, \qquad u \mapsto \frac{\int_\Omega \nabla u \cdot A_t \nabla u \, dx}{\left(\int_\Omega |u|^q J_t \, dx\right)^{2/q}},$$

with $J_t = |\det D_x \Phi|$ and $A_t = J_t (D_x\Phi)^{-1}((D_x\Phi)^{-1})^\top$.

Then, with $v_t = u_{q,\Phi(t,\Omega)} \circ \Phi(t,\cdot) \in H^1_0(\Omega)$,

$$\lambda_q(\Phi(t,\Omega)) = \mathcal{F}_t(v_t) = \inf_{u \in H^1_0(\Omega)\setminus\{0\}} \mathcal{F}_t(u).$$

Therefore

$$\lambda_q(\Omega) \leq \mathcal{F}_0(v_t) \qquad \text{and} \qquad \lambda_q(\Phi(t,\Omega)) \leq \mathcal{F}_t(v_0)\,.$$

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Therefore

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Furthermore, uniformly in compact subsets of $H_0^1(\Omega) \setminus \{0\}$

$$\mathcal{F}_t = \mathcal{F}_0 + t\dot{\mathcal{F}} + o(t).$$

and hence

$$\dot{\mathcal{F}}(v_t) + o(1) \leq \frac{\lambda_q(\Phi(t,\Omega)) - \lambda_q(\Omega)}{t} \leq \dot{\mathcal{F}}(v_0) + o(1)\,.$$

Approximation of Minkowski sum

Remaining problem: Want to construct regular map Φ so that, for $t > 0$ small, $\Phi(t, \Omega)$ approximates $\Omega + tB$,

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Define the **signed distance function**

$$\delta_\Omega(x) = \mathrm{dist}(x, \Omega) - \mathrm{dist}(x, \Omega^c), \quad \text{note that } |\nabla \delta_\Omega| = 1 \text{ a.e.}$$

Then, for $t > 0$,

$$\Omega + tB = \{x \in \mathbb{R}^d : \delta_\Omega(x) < t\}$$

and a natural candidate for Φ is

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But if $\partial\Omega$ is non-regular then so is this map.

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$$\Omega + tB = \{x \in \mathbb{R}^d : \delta_\Omega(x) < t\}$$

and a natural candidate for Φ is

$$(t, x) \mapsto x + t\nabla\delta_\Omega(x).$$

But if $\partial\Omega$ is non-regular then so is this map.

Solution: Replace $\nabla\delta_\Omega$ by a new vector field obtained by localizing $\nabla\delta_\Omega$ close to $\partial\Omega$ and mollifying.

Approximation of Minkowski sum

Theorem

Let $\Omega \subset \mathbb{R}^d$ *be open and bounded with* C^1 *boundary and fix* $\varepsilon, \delta > 0$*. There exists a map* $\Phi \in C^1((-1,1); C^\infty(\mathbb{R}^d; \mathbb{R}^d))$ *so that*

$$\Phi(t,x) = x + t\dot{\Phi}(x) + o_{t\to 0}(t) \quad \text{in } W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)$$

and

- *for* $|t|$ *sufficiently small* $\Phi(t,\cdot)$ *is a diffeomorphism of* $\mathbb{R}^d$ *onto itself,*
- *for sufficiently small* $t > 0$*,*

$$\Phi(t,\Omega) \subseteq \Omega + tB \subset \Phi((1+\delta)t,\Omega)$$

- *and* $\|\dot{\Phi} - \nu_{\partial\Omega}\|_{L^\infty(\partial\Omega)} < \varepsilon$*.*

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Remark: The assumptions are essentially sharp: Setting

$$\rho(\Omega) := \inf\{\|X - \nu_{\partial\Omega}\|_{L^\infty(\partial\Omega)} : X \in C^0(\partial\Omega;\mathbb{R}^d), |X| = 1\}$$

then by Hofmann–Mitrea–Taylor '07

$$\rho(\Omega) = 0 \quad \iff \quad \partial\Omega \text{ is } C^1,$$
$$\rho(\Omega) < \sqrt{2} \quad \iff \quad \partial\Omega \text{ is Lipschitz.}$$

Thank you for your attention!